Abstract:
This paper works on the inference independent clustering method to solve the problem of classifying a classical set given a fuzzy set and a T-indistinguishability. It is shown an application of this method for computing measures of specificity of fuzzy sets under T-indistinguishabilities.

Keywords: T-indistinguishability, inference independent classes, clustering.

1 INTRODUCTION
This paper gives a method to classify in classes a finite set $X$ given a fuzzy set and a T-indistinguishability on $X$.

The new method builds up a set of classes of $X$ that are ‘inference independent’, that is, a set of classes in such a way that given two elements of $X$ in different classes, it is not possible to infer a degree of membership of an element greater than the degree of membership given by $\mu$ by fuzzy inference through the t-norm $T$ and the T-indistinguishability with the other element.

It is also given a fuzzy set on the set of inference independent classes.

It is shown an application of the method to compute measures of specificity of fuzzy sets under T-indistinguishabilities. When the knowledge available is increased through a T-indistinguishability, the specificity of fuzzy sets is also increased. The specificity of a fuzzy set under a T-indistinguishability can be computed as the specificity of the before defined fuzzy set.

$X$ will be a crisp finite set, and $R: X \times X \to [0, 1]$ a T-indistinguishability (that is, $R$ is reflexive, symmetric and T-transitive).

A T-indistinguishability is called a similarity when $T = \min$.

2 INFERENCE INDEPENDENT CLASSES
Let $\mu$ be a fuzzy set on a finite space $X = \{x_1, ..., x_n\}$, let $S$ be a T-indistinguishability on $X$ and let $T$ be a t-norm.

Definition
$x_k$ is related with $x_j$, and it is denoted by $x_k \triangleright x_j$, if and only if $T(\mu(x_k), S(x_k, x_j)) \geq \mu(x_j)$.

Two elements $x_k$ and $x_j$ in $X$ are in the same inference independent class if and only if they are comparable by the $\geq$ preorder.

So, it is defined the class of an element $x_k$ as follows:

$[x_k] = \{x_j \text{ such that } x_k \triangleright x_j \text{ or } x_j \triangleright x_k\}$

This definition means that when $x_k$ is related with $x_j$, it is possible to deduce the same or more of what we know of $x_j$ from $x_k$ by making fuzzy inference with the t-norm $T$ and the T-indistinguishability with the other element.

It is also given a fuzzy set on the set of inference independent classes.

Proposition
Let $\mu$ be a fuzzy set on a finite set $X=\{x_1, ..., x_n\}$ and let $S$ be a T-indistinguishability, then the relation $\geq$ is a classical preorder relation on $X$. 
The \( \geq \) relation is reflexive: 
\[
T(\mu(x_i), S(x_i, x_i)) = T(\mu(x_i), 1) = \mu(x_i), \text{ so } x_i \geq x_i.
\]
The \( \geq \) relation is transitive:
Let's suppose that \( x_i \geq x_j \) and \( x_j \geq x_k \).
\[
x_i \geq x_j, \text{ so } T(\mu(x_i), S(x_i, x_j)) \geq \mu(x_j).
\]
\[
x_j \geq x_k, \text{ so } T(\mu(x_j), S(x_j, x_k)) \geq \mu(x_k).
\]
Hence \( \mu(x_k) \leq T(\mu(x_i), S(x_i, x_j), S(x_j, x_k)) \leq T(\mu(x_i), T(S(x_i, x_j), S(x_j, x_k))) \) (\( T \) is associative).
\[
= T(\mu(x_i), T(S(x_i, x_j), S(x_j, x_k))) \leq T(\mu(x_i), S(x_j, x_k)) \leq T(\mu(x_i), S(x_i, x_k)) \text{ (} S \text{ is } T\text{-transitive)}
\]
and so \( x_i \geq x_k. \]

Definition
The fuzzy set \( \mathfrak{I} \) on the crisp set of inference independent classes is defined as follows:
\[
\mathfrak{I}([x_k]) = \text{Max}_j(T(\mu(x_j), S(x_k, x_j))).
\]

It is trivial to show that the membership degree of \( [x_k] \) in \( \mathfrak{I} \) is greater or equal than \( \mu(x_k) \) for all \( x_k \) in X.

3 ALGORITHM TO COMPUTE INFERENCE INDEPENDENT CLASSES

By the previous definition, the membership degree of the classes of the elements \( \{x_1, \ldots, x_n\} \) is computed by the Max-T rule of compositional inference using the fuzzy set \( \mu \) and the T-indistinguishability \( S \).

So, \( \mathfrak{I}([x_j]) = \text{Max}_k(T(\mu(x_k), S(x_k, x_j))) \)
\[
= T(\mu(x_k), S(x_k, x_j)) \text{ for a particular } k.
\]
If \( k \neq j \), then \( x_k \) represents the class of \( x_j \) (\( x_k \geq x_j \)).
The following algorithm's purpose is to get rid of elements \( x_k \) when it exits a \( k \) such that \( [x_k] = [x_j] \) because \( x_k \geq x_j \).
The transitive property of the \( \geq \) relation is necessary for this algorithm to finish in a few steps, because when an element \( x_k \) represents another element \( x_j \) (\( x_k \geq x_j \)), we can eliminate \( x_j \) without taking care that \( x_j \) could represent a third element \( x_i \) (\( x_i \geq x_j \)), because in this case \( x_k \) would represent \( x_i \) (\( x_k \geq x_i \)) and \( x_i \), \( x_j \) and \( x_k \) would belong to \( [x_k] \). As \( x_k \) would represent \( x_i \), the algorithm also gets rid of \( x_j \) in a further step.

In summary, the algorithm detects and eliminates the elements of X that are represented by other elements, toward getting a final set of elements X' that represents the different inference independent classes \( (X' \subseteq X) \).

This algorithm steps are the following:

Step 1: Compute \( \mu \circ_{\text{Max-T}} S(\ast, x_1) \).

If \( \text{Max}_j(T(\mu(x_j), S(x_j, x_1))) \geq \mu(x_1) \) for some \( j \neq 1 \) then \( X^1 = X - \{x_1\} \), and \( \mu^1 \) and \( S^1 \) are the restrictions of \( \mu \) and \( S \) to \( X^1 \).

Otherwise, \( X^1 = X \) (\( x_1 \) represents its own class and will belong to the final set of classes \( X' \)).

Step k: Compute \( \mu \circ_{\text{Max-T}} S(\ast, x_k) \).

If \( \text{Max}_j(T(\mu^k(x_j), S^k(x_j, x_k))) \geq \mu^k(x_k) \) for some \( j \neq k \) then \( X^k = X^{k-1} - \{x_k\} \).

Otherwise \( X^k = X^{k-1} \).

Repeating this process until the \( n^{th} \) step, the set \( X^n = X' \) is the set of inference independent classes and their membership degree are given by the fuzzy set restricted to \( X^n \).

Example
Let \( \mu \) be the fuzzy set on \( X = \{x_1, \ldots, x_3\} \):
\[
\mu = 1/x_1 + 0.7/x_2 + 0.5/x_3 + 0.2/x_4 + 0/x_5.
\]
Let \( S \) be a T-Indistinguishability represented by
\[
S = \begin{pmatrix}
1 & 1 & 0 & 0.5 & 0.2 \\
1 & 1 & 0 & 0.5 & 0.2 \\
0 & 0 & 1 & 0 & 0 \\
0.5 & 0.5 & 0 & 1 & 0.2 \\
0.2 & 0.2 & 0 & 0.2 & 1
\end{pmatrix},
\]
which is reflexive and Min-transitive.
Let \( T \) be the t-norm minimum. The membership degree \( \mathfrak{I} \) of the inference independent classes for the elements \( x_i \) are the following:
The following algorithm steps are done to decide a set of classes that are Min-inference independent.

**Step 1** Compute
\[ \mu o_{\text{Max-Min}} S(*, x_1) = \]
\[
\begin{pmatrix}
1 \\
1 \\
0.5 \\
0.2
\end{pmatrix}
\]

\[ = (1, 0.7, 0.5, 0.2, 0) o_{\text{Max-Min}} S(*, x_1) = (1, 1, 0.5, 0.5, 0.2) \]

As \( 1 = \text{Max}(T(\mu(x_1), S(x_1, x_1))) = \mu(x_1) \), then \( x_1 \) represents its own class, and \([x_1] \) belongs to \(X / \geq\).

So \(X^1 = X, \mu^1 = \mu\) and \(S^1 = S\).

**Step 2** Compute
\[ \mu^1 o_{\text{Max-T}} S^1(*, x_2) = \]
\[
\begin{pmatrix}
1 \\
1 \\
0.5 \\
0.2
\end{pmatrix}
\]

\[ = \text{Max}\{1, 0.7, 0.5, 0.2, 0\} = 1 = \mu(x_2). \]

As \( 1 = \text{Max}(T(\mu^1(x_1), S^1(x_1, x_2))) = \mu^1(x_2) \) then \( x_2 \) represents \( x_2 \) (by the relation \( \geq \)) that is, \([x_1] = [x_2] \), so \(X^2 = X^1 - \{x_2\} \).

**Step 3** Compute
\[ \mu^2 o_{\text{Max-T}} S^2(*, x_3) = \]
\[
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

\[ = \text{Max}\{0, 0.5, 0.2, 0\} = 0.5 = \mu(x_3). \]

As \( 0.5 = \mu(x_3), \ x_3 \) represents its own class and \([x_3] \) will be a new element in \(X / \geq\).

So \(X^3 = X^2, \mu^3 = \mu^2 \) and \(S^3 = S^2\).

**Step 4** Compute
\[ \mu^3 o_{\text{Max-T}} S^3(*, x_4) = \]
\[
\begin{pmatrix}
0.5 \\
0 \\
1 \\
0.2
\end{pmatrix}
\]

\[ = \text{Max}\{0.5, 0.2, 0\} = 0.5 \geq 0.2 = \mu(x_4). \]

As \( 0.5 = \text{Max}(T(\mu^3(x_1), S^3(x_1, x_4))) = \mu^3(x_4) \), and \(0.2\) then \( x_4 \) represents \( x_4 \) by the relation \( \geq \).,

So \(X^4 = X^3 - \{x_4\} \).

\[ \mu^4 is \mu^3 \text{ restricted to } X^4 \text{ and } \]
\[ S^4 = \]
\[
\begin{pmatrix}
1 & 0 & 0.2 \\
0 & 1 & 0 \\
0.2 & 0 & 1
\end{pmatrix}
\]
on $X^4 \times X^4$.

**Step 5** computes

$$\mu^4 \circ_{\text{Max-T}} S^4(*, x_5) =$$

$$\begin{pmatrix} 0.2 \\ 0 \\ 1 \end{pmatrix}$$

Max{0.2, 0, 0} = 0.2 ≥ 0 = $\mu(x_5)$.

As 0.2 = Max(T($\mu^3(x_1)$, $S^4(x_1, x_5)$)) ≥ $\mu^4(x_5)$ = 0 then $x_1$ represents $x_5$ by the relation $\geq$.

$X^5 = X^4 - \{x_5\} = \{x_1, x_3\}$, so the set of Min-inference independent classes in $X / \geq$ are \{\{x_1\}, [x_3]\} = \{\{x_1, x_2, x_4, x_5\}, [x_3]\}, and their membership degree are those of $\mathbb{I}$ restricted to $X^5$, that is, {1/$x_1$, 0.5/$x_3$}.

4 APPLYING THE ALGORITHM TO COMPUTE SOME MEASURES OF SPECIFICITY.


The $\alpha$-cut of a similarity $S$ is a classical equivalence relation [3] denoted $S_\alpha$. Let $\pi_\alpha$ be the set of equivalence classes of $S$ for a given value $\alpha$. Let $\mu_\alpha$/$S$ be the set of equivalence classes of $\pi_\alpha$ defined in the following way: class $\pi_\alpha(i)$ belongs to $\mu_\alpha$/$S$ if there exists an element $x$ contained in $\pi_\alpha(i)$ and in the $\mu$’s $\alpha$-cut ($\mu_\alpha$).

**Definition**

Yager [1991, 91] definition of measure of specificity of a fuzzy set $\mu$ under a similarity is the following:

$$S_p(\mu/S) = \int_0^{\alpha_{\max}} \frac{1}{\text{Card}(\mu_\alpha/S)} d\alpha.$$  

The measure of specificity under similarities are maximal when $\mu_\alpha$ is contained in one class of $S_\alpha$ for all $\alpha$.

This definition is good enough when the information is increased by a similarity, but it is not well defined for any T-indistinguishability, because when $T$ is not the minimum t-norm the $\alpha$-cut of $S$ is not an equivalence relation and then $\mu_\alpha$/$S$ is not well defined.

**Definition**

Let $Sp$ be a measure of specificity.

A measure of specificity of a fuzzy set $\mu$ under a T-indistinguishability $S$ is the measure of specificity $Sp$ of the fuzzy set $\mathbb{I}$ on the set of classes $X^0 = X / \geq$.

**Theorem**

The measure of specificity of $\mu$ under $S$ computed by the algorithm satisfies the four axioms of a measure of specificity under a T-indistinguishability.

**Proof**

The proof is in [1].

**EXAMPLE**

When using the previous example given to show how the algorithm works, the following Min-inference independent classes are found: \{\{x_1, x_2, x_4, x_5\}, [x_3]\}. Their membership degrees to the fuzzy set $\mathbb{I}$ are $1/|x_1|+0.5/|x_3|$.

So, the measure of specificity of $\mu$ under the Min-indistinguishability $S$ is the measure of specificity of the fuzzy set $\mathbb{I}$ on the set of classes $[x_1]$ and $[x_3]$ with membership degrees $1/|x_1|+0.5/|x_3|$.

When using, for example, the linear measure of specificity of Yager [1990] with a weight $w_2 = 1$, the measure of specificity of $\mu$ under $S$ is:

$$Sp (\mu / S) = Sp( \mathbb{I} ) = Sp (1/|x_1|+0.5/|x_3|) = 1 - 0.5 = 0.5.$$  

Compare this result with the linear measure of specificity of $\mu$ with a weight $w_2 = 1$.

$$Sp (\mu) = Sp (1/|x_1|+0.7/|x_2|+0.5/|x_3|+0.2/|x_4|+0/|x_5|)$$

$$= 1 - 0.7 = 0.3.$$
Observe that the measure of specificity of $\mu$ under a T-indistinguishability $S$ is greater than the measure of specificity of $\mu$. This is because the T-indistinguishability adds information which tells that four of the five elements of $X$ are similar. When using the measure of specificity as a measure of the information of a fuzzy set in order to make a decision of an element of $X$, the Min-indistinguishability is telling us that four of the five possible decisions are similar, so the decision is simplify to two classes of elements of $X$. The fuzzy set $\mathcal{I}$ on the inference independent classes is useful to define and compute new measures of specificity of a fuzzy set $\mu$ when the information is increased by a T-indistinguishability.

Conclusions
This paper gives an algorithm to classify a finite set $X$ into inference independent classes given a fuzzy set and a T-indistinguishability on $X$. A fuzzy set $\mathcal{I}$ on the set of inference independent classes is given. It is shown an application of the algorithm to define and compute new measures of specificity of a fuzzy set under a T-indistinguishability.

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References


