GENERAL MEASURES OF SPECIFICITY OF FUZZY SETS UNDER T-INDISTINGUISHABILITIES.

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Abstract:

This paper extends the concept of measures of specificity of fuzzy sets under similarities to measures of specificity under T-indistinguishabilities, giving an axiomatic definitions and some examples using an algorithmic method to compute them.

Keywords: Measure of specificity, T-indistinguishability.

1 INTRODUCTION

The measures of specificity under similarities [R. R. Yager, 1991] are based on the idea that when the knowledge available is increased through a similarity, the specificity of a fuzzy sets is also increased. This idea can also be applied to any T-indistinguishability using new methods.

Given a fuzzy set \( \mu \) on a finite space \( X \) and a similarity relation, the classes of \( X \) can be defined for each value \( \alpha \) in \([0, 1]\) because the \( \alpha \)-cuts of a similarity are classical equivalence relations. However the \( \alpha \)-cuts of a T-indistinguishability are not necessarily a classical equivalence relations, so we need to introduce new concepts, definitions and methods.

There are proposed some axioms that characterise measures of specificity under T-indistinguishabilities, generalising the previous concepts on measures of specificity under similarities.

It is given an algorithmic method to compute a measures of specificity under T-indistinguishabilities and it is proven that this definition verify the axioms of measures of specificity under T-indistinguishabilities.

2 PRELIMINARIES

Definition 2.1: A fuzzy set \( \mu \) on \( X \) is normal if there exits an element \( x_1 \in X \) such that \( \mu(x_1) = a_1 = 1 \).

Definition 2.2: Measure of specificity

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Let $X$ be a set with elements $\{x_i\}$ and let $[0, 1]^X$ be the class of fuzzy sets of $X$. A measure of specificity [R. R. Yager; 1990] is a function $S_p: [0, 1]^X \rightarrow [0, 1]$ such that:

1. $S_p(\mu) = 1$ if and only if $\mu$ is a singleton ($\mu = \{x_1\}$).
2. $S_p(\emptyset) = 0$
3. If $\mu$ and $\eta$ are normal fuzzy sets in $X$ and $\mu \subset \eta$, then $S_p(\mu) \geq S_p(\eta)$.

The first condition imposes that the specificity is one (maximum value) only for crisp sets with just one element (singletons). The second condition assumes the minimum specificity for the null set. Other non-null fuzzy sets could also have specificity zero. The third condition requires that the specificity measure of a normal fuzzy set decreases when the membership degree of its elements increases.

If we would have to choose one element from a set of elements, and we would have a fuzzy set with a degree of satisfaction of each element, it is desirable to have a singleton or a high specificity fuzzy set to be sure that our election is right.


The introduction of this concept is made through the ‘Yager’s jacket problem’: If we know that the weather is over 20 °C, this information itself is not very specific, but it is indeed specific in the problem of choosing the right jacket to wear. Some degree ranges could become similar or indistinguishable in order to make a decision then our information is increased thought a T-indistinguishability.

A fuzzy relation is a T-indistinguishability when it is reflexive, symmetric and T-transitive. When $T$ is the t-norm Minimum, then the Min-indistinguishability is called similarity.

The $\alpha$-cut of a similarity $S$ is a classic equivalence relation denoted $S_\alpha$.

Let $\pi_\alpha$ be the set of equivalence classes of $S$ for a given value $\alpha$. Let $\mu_\alpha/S$ be the set of equivalence classes of $\pi_\alpha$ defined in the following way: class $\pi_\alpha(i)$ belongs to $\mu_\alpha/S$ if there exists an element $x$ contained in $\pi_\alpha(i)$ and in the $\mu$ $\alpha$-cut ($\mu_\alpha$).

**Definition 2.3:**

R. R. Yager [1991, 91] defines the measure of specificity of a fuzzy set $\mu$ under a similarity as follows:

$$S_p(\mu/S) = \frac{\alpha_{\max}}{\int_0^{\alpha_{\max}} \frac{1}{\text{Card}(\mu_\alpha/S)} \, d\alpha.}$$

The measure of specificity under similarities are maximal when $\mu_\alpha$ is contained in one class of $S_\alpha$ for all $\alpha$.

This definition is good when the information is increased by a similarity, but it is not well defined for any T-indistinguishability, because when $T$ is not the minimum t-norm the $\alpha$-cut of $S$ is not an equivalence relation and then $\mu_\alpha/S$ is not well defined.
3 AXIOMS OF MEASURES OF SPECIFICITY UNDER A T-INDISTINGUISHABILITY

Let \( X \) be a crisp finite set, let \( \mu \) be a fuzzy set or a possibility distribution on \( X \) and let \( S: X \times X \to [0, 1] \) be a T-indistinguishability.

**Definition 3.1:**

Given a measure of specificity \( Sp \) on fuzzy sets and given a T-indistinguishability \( S \), a measure \( Sp(\mu / S) \) is a measure of specificity under T-indistinguishabilities when it verify the following four axioms:

1. \( Sp(\{x\} / S) = 1 \)
2. \( Sp(\emptyset / S) = 0 \)
3. \( Sp(\mu / \text{Id}) = Sp(\mu) \)
4. \( Sp(\mu / S) \geq Sp(\mu) \)

The first axiom lies on the concept that the specificity of a singleton is always one, even when the information available is increased through a T-indistinguishability. In the case when the indistinguishability is a classical equivalence relation, and the specificity measure is used as a measure of tranquility in a decision making process, then if the input fuzzy set is a singleton we could easily choose the class of the element \( x \).

The second axiom is similar to the second axiom of measure of specificity [R. R. Yager; 1990], because then the fuzzy set is the empty set, it does not provide any information, even if we know a T-indistinguishability, so the specificity should be zero.

The third axiom stands that when the T-indistinguishability is the Identity, then the specificity of the fuzzy set under \( S = \text{Id} \) is the specificity of the fuzzy set. In this case \( S = \text{Id} \) is classical equivalence relation that gives a class for each element in \( X \), so the indistinguishability do not provide extra information on the one provided in the fuzzy set.

The fourth axiom indicate that when the information available is increased through a indistinguishability, the usability of the information contained in the fuzzy set is always increased or keeped. Thinking in the case of a classical equivalence relation, it could be less or equal number of classes to choose, so a decision could be easier and so the specificity must be higher or equal.

4 ALGORITHM TO COMPUTE THE INFERRENCE INDEPENDENT SET.

Let \( \mu \) be a fuzzy set on a finite space \( X = \{x_1, \ldots, x_n\} \), let \( T \) be a t-norm and let \( S \) be a T-indistinguishability on \( X \).

**Definition 4.1:** the ‘infers’ relation \( \geq \)

\( x_k \) is related with \( x_j \) (and it is denoted by \( x_k \geq x_j \)) if and only if \( T(\mu(x_k), S(x_k, x_j)) \geq \mu(x_j) \).
This definition means that when \( x_k \) is related with \( x_j \), in monotonic reasoning it is possible to deduce the same or more information of what we know from \( x_k \) of \( x_j \) by making fuzzy inference with the t-norm \( T \) and the T-indistinguishability. That is, \( x_k \) is related with \( x_j \) when the information on \( x_j \) is increased by knowing the membership degree of \( x_k \) and its T-indistinguishability relation with \( x_j \).

**Proposition 4.1:**

Let \( \mu \) be a fuzzy set on a finite set \( X=\{x_1, \ldots, x_n\} \) and let \( S \) be a T-indistinguishability, then the relation \( \geq \) is a classic preorder relation on \( X \).

**Proof**

The \( \geq \) relation is reflexive:

\[
T(\mu(x_i), S(x_i, x_i)) = T(\mu(x_i), 1) = \mu(x_i), \text{ so } x_i \geq x_i.
\]

The \( \geq \) relation is transitive:

Let’s suppose that \( x_i \geq x_j \) and \( x_j \geq x_k \).

\[
\begin{align*}
x_i \geq x_j, & \text{ so } T(\mu(x_i), S(x_i, x_j)) \geq \mu(x_j). \\
x_j \geq x_k, & \text{ so } T(\mu(x_j), S(x_j, x_k)) \geq \mu(x_k).
\end{align*}
\]

Hence \( \mu(x_k) \leq T(\mu(x_j), S(x_j, x_k)) \)

\[
\begin{align*}
& \leq T(T(\mu(x_i), S(x_i, x_j)), S(x_j, x_k)) \quad (T \text{ is associative}) \\
& = T(\mu(x_i), T(S(x_i, x_j), S(x_j, x_k))) \quad (S \text{ is T-transitive}) \\
& \leq T(\mu(x_i), S(x_i, x_k))
\end{align*}
\]

and so \( x_i \geq x_k \). \( \blacksquare \)

**Algorithm to compute de inference independent set**

It is given an algorithm that finds a subset of elements of \( X \) that can not be inferred to each other by making fuzzy inference with the t-norm \( T \), the fuzzy set membership degrees and the T-indistinguishability degrees. This is going to be called the **inference independent set** \( X' \). None of the elements of \( X' \) will be related by the relation \( \geq \) with another element of \( X' \).

The algorithm to compute the inference independent set is defined as follows:
Step 1: Compute $\mu \circ_{\text{Max-T}} S(\ast, x_1)$.

If $\max_j(T(\mu(x_j), S(x_j, x_1))) \geq \mu(x_1)$ for some $j \neq 1$ then $X^1 := X - \{x_1\}$, and $\mu^1$ and $S^1$ are the restrictions of de $\mu$ and $S$ to $X^1$.

Otherwise, $X^1 = X$ ($x_1$ becomes an element of the inference independent set $X'$).

Step k: Compute $\mu \circ_{\text{Max-T}} S(\ast, x_k)$.

If $\max_j(T(\mu^{k-1}(x_j), S^{k-1}(x_j, x_k))) \geq \mu^{k-1}(x_k)$ for some $j \neq k$, then $X^k := X^{k-1} - \{x_k\}$.

Otherwise $X^k = X^{k-1}$.

Repeating this process until the $n^{\text{th}}$ step, let define $X' = X^n$.

Definition 4.2:

The fuzzy set $\Im$ on $X$ is defined as follows:

If $x_k \in X'$ then $\Im(x_k) = \mu(x_k)$ else $\Im(x_k) = 0$.

Note that $\Im$ is defined on $X$, but could be called a fuzzy set on the inference independent set because the non inference independent elements in $X$ (that can also be called the inferred elements) have membership degree zero.

The algorithm purpose is to get rid of elements $x_j$ when it exits a $k$ such that $x_k \geq x_j$. For those elements we will give $\Im(x_k) = 0$ because those elements belong to $X - X'$.

The transitive property of the $\geq$ relation is necessary for this algorithm to finish in a few steps, because when an element $x_k$ is related with another element $x_j$ ($x_k \geq x_j$), we can eliminate $x_j$ without taking care that $x_j$ could be related with a third element $x_i$ ($x_j \geq x_i$), because in this case $x_k$ would be related with $x_i$ ($x_k \geq x_i$). As $x_k$ would be related with $x_i$, the algorithm also gets rid of $x_i$ in a further step.

In summary, the algorithm detects and eliminates the elements of $X$ that are related with other elements, toward getting a final set of elements $X'$, the inference independent set ($X' \subseteq X$), that will be used to define the fuzzy set $\Im$.

Example 4.1

Let $\mu$ be a fuzzy set on $X = \{x_1, \ldots, x_5\}$: $\mu = 1/x_1 + 0.7/x_2 + 0.5/x_3 + 0.2/x_4 + 0/x_5$.

Let $S$ be a Min-Indistinguishability represented by
which is reflexive and Min-transitive.

Let $T$ be the t-norm minimum.

The following algorithm steps are done to decide a Min-inference independent set.

**Step 1** computes

$$\mu \circ_{\text{Max-Min}} S(\cdot, x_1) = (1, 0.7, 0.5, 0.2, 0) \circ_{\text{Max-Min}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0.5 \\ 0.2 \end{pmatrix} = \text{Max}\{1, 0.7, 0.5, 0.2, 0\} = 1 = \mu(x_1).$$

As $1 = \text{Max}(T(\mu(x_1), S(x_1, x_1))) = \mu(x_1)$, then $x_1$ belongs to $X_1$. So $X_1 = X$, $\mu^1 = \mu$, $S^1 = S$, and $x_1$ will be a new element in $X'$ (it is not Min-$S$ inferred by other element).

**Step 2** computes

$$\mu^1 \circ_{\text{Max-T}} S^1(\cdot, x_2) = (1, 0.7, 0.5, 0.2, 0) \circ_{\text{Max-Min}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0.5 \\ 0.2 \end{pmatrix} = \text{Max}\{1, 0.7, 0.2, 0\} = 1 \geq 0.7 = \mu(x_2)$$

As $1 = \text{Max}(T(\mu^1(x_1), S^1(x_1, x_2))) \geq \mu^1(x_2)$ then $x_1$ is related with $x_2$ (by the relation $\geq$), so $X_2 = X^1 - \{x_2\} = \{x_1, x_3, x_4, x_5\}$, $\mu^2$ is $\mu^1$ restricted to $X^1$ and
$$S^2 = \begin{pmatrix} 1 & 0 & 0.5 & 0.2 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 1 & 0.2 \\ 0.2 & 0 & 0.2 & 1 \end{pmatrix}$$

on $X^2 \times X^2$, (it is, on $\{x_1, x_3, x_4, x_5\}^2$). In summary, $x_2$ will not be a new element in $X'$.

**Step 3** computes

$$\mu^2 \circ_{\max-T} S^2(\bullet, x_3) = (1, 0.5, 0.2, 0) \circ_{\max-min} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \max\{0, 0.5, 0, 0\} = 0.5 = \mu(x_3).$$

As $0.5 = \mu(x_3)$, $x_3$ will be a new element in $X'$, $X^3 = X^2$, $\mu^3 = \mu^2$ and $S^3 = S^2$.

**Step 4** computes

$$\mu^3 \circ_{\max-T} S^3(\bullet, x_4) = (1, 0.5, 0.2, 0) \circ_{\max-min} \begin{pmatrix} 0.5 \\ 0 \\ 1 \\ 0.2 \end{pmatrix} = \max\{0.5, 0, 0.2, 0\} = 0.5 \geq 0.2 = \mu(x_4).$$

As $0.5 = \max(T(\mu^3(x_1), S^3(x_1, x_4))) \geq \mu^3(x_4) = 0.2$ then $x_1$ is related with $x_4$ by the relation $\geq$, so $X^4 = X^3 - \{x_4\} = \{x_1, x_3, x_5\}$, $\mu^4$ is $\mu^3$ restricted to $X^4$ and

$$S^4 = \begin{pmatrix} 1 & 0 & 0.2 \\ 0 & 1 & 0 \\ 0.2 & 0 & 1 \end{pmatrix}$$

on $X^4 \times X^4$. In summary, $x_4$ will not be a new element in $X'$.

**Step 5** computes
\[
\mu^4 \circ_{\text{Max-T}} S^4^\ast (x_5) = (1, 0.5, 0) \circ_{\text{Max-Min}} \begin{pmatrix}
0.2 \\
0 \\
1
\end{pmatrix} = \text{Max}\{0.2, 0, 0\} = 0.2 \geq 0 = \mu(x_5).
\]

As 0.2 = Max(T(\mu^3(x_1), S^4(x_1, x_5))) \geq \mu^4(x_5) = 0 then \(x_1\) is related with \(x_5\) by the relation \(\geq\).

\(X^5 = X^4^\ast \{x_5\} = \{x_1, x_3\}\), so Min-inference independent set is \(X' = \{x_1, x_3\}\) and the fuzzy set \(\mathcal{I}\) is \(\{1/x_1, 0/x_2, 0.5/x_3, 0/x_4, 0/x_5\}\).

**Example 4.2:**

Let \(T\) be the t-norm Product.

Let \(\mu\) be a fuzzy set on \(X=\{x_1, \ldots, x_5\}\): \(\mu = 1/x_1 + 0.7/x_2 + 0.5/x_3 + 0.2/x_4 + 0/x_5\).

Let \(S\) be a Prod-Indistinguishability represented by

\[
S = \begin{pmatrix}
1 & 0.25 & 0 & 0.5 & 0.2 \\
0.25 & 1 & 0 & 0.5 & 0.2 \\
0 & 0 & 1 & 0 & 0 \\
0.5 & 0.5 & 0 & 1 & 0.1 \\
0.2 & 0.2 & 0 & 0.1 & 1
\end{pmatrix},
\]

which is reflexive and Prod-transitive.

Note that \(S\) is not a similarity, because \(S(x_1, x_2) = 0.25 < \text{Min}\{S(x_1, x_4), S(x_4, x_2)\} = \text{Min}\{0.5, 0.5\} = 0.5\).

The following algorithm steps are done to decide a Prod-inference independent set.

**Step 1** computes

\[
\mu_{\text{Max-Prod}} S^\ast (x_1) = (1, 0.7, 0.5, 0.2, 0) \circ_{\text{Max-Prod}} \begin{pmatrix}
1 \\
0.25 \\
0 \\
0.5 \\
0.2
\end{pmatrix} = \text{Max}\{1, 0.175, 0, 0.1, 0\} = 1 = \mu(x_1)
\]

As 1 = Max(T(\mu(x_1), S(x_1, x_1))) = \mu(x_1), then \(x_1\) belongs to \(X^l\). So \(X^l = X, \mu^l = \mu, S^l = S\) and \(x_1\) will be an element in \(X^l\) (\(x_1\) is not Prod-S inferred by other element).
Step 2 computes

\[ \mu_1 \circ \text{Max-T} \circ S^1(*, x_2) = (1, 0.7, 0.5, 0.2, 0) \circ \text{Max-Prod} \begin{pmatrix} 0.25 \\ 1 \\ 0 \\ 0.5 \\ 0.2 \end{pmatrix} = \text{Max}\{0.25, 0.7, 0, 0.1, 0\} = 0.7 = \mu(x_2) \]

As 0.7 = Max(T(\mu(x_2), S(x_2, x_2))) = \mu(x_2), then \( x_2 \) belongs to \( X^2 \). So \( X^2 = X^1, \mu_2 = \mu_1, S^2 = S^1 \) and \( x_2 \) will be an element in \( X' \).

Step 3 computes

\[ \mu_2 \circ \text{Max-T} \circ S^2(*, x_3) = (1, 0.7, 0.5, 0.2, 0) \circ \text{Max-Prod} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \text{Max}\{0, 0, 0.5, 0, 0\} = 0.5 = \mu(x_3). \]

As 0.5 = \( \mu(x_3) \), \( x_3 \) will be an element in \( X' \), \( X^3 = X^2, \mu_3 = \mu_2 \) and \( S^3 = S^2 \). So \( x_3 \) will be an element in \( X' \).

Step 4 computes

\[ \mu_3 \circ \text{Max-T} \circ S^3(*, x_4) = (1, 0.7, 0.5, 0.2, 0) \circ \text{Max-Prod} \begin{pmatrix} 0.5 \\ 0.5 \\ 0 \\ 1 \\ 0.1 \end{pmatrix} = \text{Max}\{0.5, 0.35, 0, 0.2, 0\} = 0.5 \geq 0.2 = \mu(x_4). \]

As 0.5 = Max(T(\mu^3(x_1), S^3(x_1, x_4))) \geq \mu^3(x_4) = 0.2 then \( x_1 \) is related with \( x_4 \) by the relation \( \geq \), so \( X^4 = X^3 - \{x_4\} = \{x_1, x_2, x_3, x_5\}, \mu^4 \) is \( \mu^3 \) restricted to \( X^4 \) and

\[ S^4 = \begin{pmatrix} 1 & 0.25 & 0 & 0.2 \\ 0.25 & 1 & 0 & 0.2 \\ 0 & 0 & 1 & 0 \\ 0.2 & 0.2 & 0 & 1 \end{pmatrix} \]
on $X^4 \times X^4$. So $x_4$ will not be an element in $X'$.

**Step 5** computes

$$
\mu_4 \circ_{\text{Max-T}} S^4(*, x_5) = (1, 0, 0, 0) \circ_{\text{Max-Prod}} \begin{pmatrix} 0.2 \\ 0.2 \\ 0 \\ 1 \end{pmatrix} = \text{Max}\{0.2, 0.14, 0, 0\} = 0.2 \geq 0 = \mu(x_5).
$$

As $0.2 = \text{Max}(T(\mu^3(x_1), S^4(x_1, x_5))) \geq \mu_4(x_5) = 0$ then $x_1$ is related with $x_5$ by the relation $\geq$. So $x_5$ will not be an element in $X'$.

$X^5 = X^4\setminus\{x_5\} = \{x_1, x_2, x_3\}$, so the Prod-inference independent set is $X' = \{x_1, x_2, x_3\}$ and the fuzzy set $\mathcal{I}$ is $\{1/x_1, 0.7/x_2, 0.5/x_3, 0/x_4, 0/x_5\}$.

5. APPLYING THE ALGORITHM TO COMPUTE MEASURES OF SPECIFICITY OF FUZZY SETS UNDER T-INDISTINGUISHABILITIES.

**Definition 5.1:**

Let $Sp$ be a measure of specificity. A **measure of specificity of a fuzzy set** $\mu$ **under a T-indistinguishability** $S$ is defined as the measure of specificity $Sp$ of the fuzzy set $\mathcal{I}$: $Sp(\mu/S) = Sp(\mathcal{I})$.

**Theorem 5.1:**

The measure of specificity of $\mu$ under $S$ computed by the proposed algorithm satisfy the four axioms of measures of specificity under a T-indistinguishability.

**Proof**

Let $\mu$ be a fuzzy set on a finite space $X = \{x_1, ..., x_n\}$ ordered by their membership degree so that $\mu(x_1) \geq ... \geq \mu(x_n)$.

**Axiom 1:** It is shown that $Sp(\{x\} / S) = 1$.

If $\mu$ is a singleton then $\mu(x_i) = 1$ y $\mu(x_j) = 0$ for all $j \neq 1$. The first step in the algorithm it is computed

$$
\mu \circ_{\text{Max-T}} S(*, x_1) = (1, 0,...,0) \circ_{\text{Max-T}} \begin{pmatrix} S(x_1, x_1) \\ S(x_2, x_1) \\ \vdots \\ S(x_n, x_1) \end{pmatrix} = T(1, S(x_1, x_1)) = T(1, 1) = 1
$$
So $x_1$ belongs to $X'$, $X^1 = X$ and $\mathcal{I}(x_1) = 1$.

In further steps, for any $i$ greater than 1, it is computed

$$\mu_{\text{omax}} S(\ast, x_i) = (1, 0, \ldots, 0)_{\text{omax}} T(1, S(x_1, x_i)) = S(x_1, x_i) \geq \mu^{i-1}(x_i) = 0$$

So $x_i$ is eliminated for all $i \neq 1$, $X^n = \{x_1\}$ and the measure of specificity of $Sp(\{x\} / S)$ is one because $\mathcal{I}$ is a singleton, and the first axiom of measures of specificity [Yager, 1990] stands that the specificity measure on a singleton is one.

**Axiom 2:** It is shown that $Sp(\emptyset / S) = 0$. In any step of the algorithm it is computed that

$$\mu_{\text{omax}} S(\ast, x_i) = (0, \ldots, 0)_{\text{omax}} T(0, S(x_i, x_i)) = 0,$$

so $X^n = X' = X$, but $\mathcal{I}(x_i) = 0$ for all $i$, so $\mathcal{I} = \emptyset$ and the specificity of $\mathcal{I}$ is zero (second axiom of measure of specificity [Yager, 1990]).

**Axiom 3:** It is shown that $Sp(\mu / \text{Id}) = Sp(\mu)$. When the T-indistinguishability $S$ is the Identity, in all the steps it is computed

$$\mu_{\text{omax}} S(\ast, x_i) = (\mu(x_1), \ldots, \mu(x_n))_{\text{omax}} T(0, \mu(x_i), 1) = \mu(x_i),$$

So all the elements $x_i$ of $X$ belong to $X'$, $X^n = X' = X$, and $\mathcal{I}$ is the set $\mu$, so $Sp(\mu / \text{Id}) = Sp(\mu)$.

**Axiom 4:** It is shown that $Sp(\mu / S) \geq Sp(\mu)$.

In the first step, as $x_1$ has the greatest membership degree then
\[ \mu \circ_{\text{Max-T}} S(x_1, x_1) = (\mu(x_1), \ldots, \mu(x_n)) \circ_{\text{Max-T}} \begin{pmatrix} S(x_1, x_1) \\ S(x_2, x_1) \\ \vdots \\ S(x_n, x_1) \end{pmatrix} = T(\mu(x_1), S(x_1, x_1)) = T(\mu(x_1), 1) = \mu(x_1). \]

So, \( x_1 \) belongs to \( X' \) and \( \mathcal{I}(x_1) = \mu(x_1). \)

However by the definition of \( \mathcal{I} \) it is known that \( \mathcal{I} \subseteq \mu \), and by the third axiom of measures of specificity [Yager, 1990] as the specificity of the membership degree (others than the maximum) decrease, the specificity increase, so \( \text{Sp}(\mu / S) = \text{Sp}(\mathcal{I}) \geq \text{Sp}(\mu). \)

Note that the measure of specificity applied to \( \mathcal{I} \) and \( \mu \) must be the same (with the same weights).

6 EXAMPLES

Example 6.1:

When using the first previous example given to show how the algorithm works, the Min-inference independent set is \( \{x_1, x_3\} \), and the membership degrees to the fuzzy set \( \mathcal{I} \) are \( 1/x_1 + 0/x_2 + 0.5/x_3 + 0/x_4 + 0/x_5. \)

So, the measure of specificity of \( \mu \) under the Min-indistinguishability \( S \) is the measure of specificity of the fuzzy set \( \mathcal{I} \).

a) When using, for example, the linear measure of specificity of Yager [1990] defined as

\[ \text{Sp}(\mu) = \mu(x_1) - \sum_{j=2}^{d} w_j \mu(x_j) \]

with a weight \( w_2 = 1 \), the measure of specificity of \( \mu \) under \( S \) is:

\[ \text{Sp}(\mu / S) = \text{Sp}(\mathcal{I}) = \text{Sp}(1/x_1 + 0/x_2 + 0.5/x_3 + 0/x_4 + 0/x_5) = 1 - 0.5 = 0.5. \]

While \( \text{Sp}(\mu) = 1 - 0.7 = 0.3 < \text{Sp}(\mu / S) \).

b) When using the measure of specificity \( \text{Sp}(\mu) = \mu(x_1) \prod_{j=2}^{d} (1 - w_j \mu(x_j)) \) with a weight \( w_2 = 1 \), the measure of specificity of \( \mu \) under \( S \) is:

\[ \text{Sp}(\mu / S) = \text{Sp}(\mathcal{I}) = \text{Sp}(1/x_1 + 0/x_2 + 0.5/x_3 + 0/x_4 + 0/x_5) = 1 \times 0.5 = 0.5. \]

While \( \text{Sp}(\mu) = 1 \times (1 - 0.7) = 0.3 < \text{Sp}(\mu / S). \)
When using the measure of specificity as a measure of the information of a fuzzy set in order to make a decision of an element of X, the Min-indistinguishability is telling us that four of the five possible decisions are similar, so the decision is simplify to two elements of X’.

Example 6.2:

In the second previous example with a Prod-Indistinguishability, it is found that Prod-inference independent set is X’ = \{x_1, x_2, x_3\} and the fuzzy set \(\mathcal{I}\) is \{1/x_1, 0.7/x_2, 0.5/x_3, 0/x_4, 0/x_5 \}.

a) When using, for example, the linear measure of specificity of Yager defined as

\[ Sp(\mu) = \mu(x_1) - \frac{d}{\sum_{j=2}^{d} w_j \mu(x_j)} \]

with a weight \(w_2 = 1\), the measure of specificity of \(\mu\) under S is:

\[ Sp(\mu / S) = Sp(\mathcal{I}) = Sp(1/x_1 + 0.7/x_2+0.5/x_3+0/x_4+0/x_5) = 1 - 0.7 = 0.3. \]

But if we choose the weights \(w_2 = 0.5, w_3 = 0.5\), the measure of specificity of \(\mu\) under S is:

\[ Sp(\mu / S) = Sp(\mathcal{I}) = Sp(1/x_1 + 0.7/x_2+0.5/x_3+0/x_4+0/x_5) = 1 - 0.35 - 0.25 = 0.4. \]

b) When using the measure of specificity \(Sp(\mu) = \mu(x_1) \prod_{j=2}^{d} (1 - w_j \mu(x_j)) \) with a weight \(w_2 = w_3 = 0.5\), the measure of specificity of \(\mu\) under S is:

\[ Sp(\mu/S) = Sp(\mathcal{I}) = Sp(1/x_1 + 0.7/x_2+0.5/x_3+0/x_4+0/x_5) = 1 \times (1-0.35) \times (1-0.25) = 1 \times 0.65 \times 0.75 = 0.4875. \]

When using the measure of specificity as a measure of the information of a fuzzy set in order to make a decision of an element of X, the Prod-indistinguishability is telling us that three of the five possible decisions are similar, so the decision is simplify to three elements of X’.

Note:

Lets compute the measure of specificity of \(\mu\) to compare with the examples results.

The highest linear measure of specificity is reached when \(w_2 = 1\). In this case

\[ Sp(\mu) = Sp(1/x_1 + 0.7/x_2+0.5/x_3+0.2/x_4+0/x_5) = \mu(x_1) - \sum_{j=2}^{d} w_j \mu(x_j) = 1 - 0.7 = 0.3. \]
Or when taking the measure of specificity $\mu(x_i) \prod_{j=2}^{d}(1 - w_j \mu(x_j))$ with $w_i = 0.5$ then

$Sp(\mu) = Sp \left( \frac{1}{x_1} + \frac{0.7}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0}{x_5} \right) = 1 \times (1 - 0.35) \times (1 - 0.25) \times (1 - 0.1) = 1 \times 0.65 \times 0.75 \times 0.9 = 0.43875$.

Observe that the measure of specificity of $\mu$ under a T-indistinguishability $S$ is greater or equal than the measure of specificity of $\mu$ when taking similar weights (it is when having a fixed measure of specificity), as axiom 4 suggest. This is because the T-indistinguishability adds information which tells that some elements of $X$ are ‘similar’ or ‘indistinguishable’ (four in the first example and three in the second example).

**Conclusions**

It is given the axioms of measures of specificity under T-indistinguishabilities.

It is shown an algorithm to find the inference independent set $X'$, and a way to find the fuzzy set $\mathcal{I}$ to compute a measure of specificity under T-indistinguishabilities as a measure of specificity of the fuzzy set $\mathcal{I}$.

It is proven that this way of computing a measure of specificity under T-indistinguishabilities satisfy the four proposed axioms of measures of specificity under T-indistinguishabilities.

**References**


