Comparing Transitive Closure with a new T-transitivization method

Luis Garmendia[1], Adela Salvador[2]

[1] Facultad de Informática, Dpto. de Lenguajes y Sistemas Informáticos, Universidad Complutense of Madrid, 28040 Madrid, Spain
lgarmend@fdi.ucm.es
[2] E.T.S.I. Caminos Canales y Puertos, Dpto. de Matemática Aplicada , Technical University of Madrid, 28040 Madrid, Spain
ma09@caminos.upm.es

Abstract. It has been developed a C++ program that generates random fuzzy relations of a given dimension and computes their T-transitive closure (that contains the initial relation) and the new T-transitivized relation (that is contained in the initial relation) for the t-norms minimum, product and Lukasiewicz. It has been computed several distances between both transitive closure and transitivized relation with the initial relation one hundred times for each dimension and for each t-norm, and the results show that the average distance of the random fuzzy relations with the transitive closure is higher than the average distance with the new transitivized relation.

1 Introduction

A new method to T-transitivize fuzzy relations [Garmendia & Salvador; 2000] can be used to give new measure of T-transitivity of fuzzy relations. It can also be used to build T-transitive fuzzy relations from a given fuzzy relation.

When the initial fuzzy relation is reflexive, the algorithm generates T-preorders that are different to the T-preorders generated form the T-transitive closure.

The transitive closure of a fuzzy relation contains the initial relation, but the transitivized relation is contained in the initial fuzzy relation.

This paper results are obtained from a C++ program that generate random fuzzy relations of a given dimension and computes their Min-transitive closure, Prod- transitive closure and W-transitive closure and their Min-transitivized relation, Prod-transitivized relation and W-transitivized relation.

It is computed the measure of T-transitivity of fuzzy relations measuring the difference between the transitivized relation and the original one, using several distances as the absolute value of the difference, euclidean distances or normalised distances. Those distances are also measured between the same random fuzzy relations and their T-transitive closures, resulting to be higher than the average distances with the T-transitivized relation for all dimensions computed.

2 Preliminaries

2 .1 The importance of transitivity

The T-transitive property is held by T-indistinguishabities and T-preorders, and it is important when making fuzzy inference to have *Tarski* consequences. The similarities and T-indistinguishabilities generalise the classical equivalence relations, and are useful to classify or to make fuzzy partitions of a set.

Even though not all the fuzzy inference in control needs transitivity, it looks important to know whether the fuzzy relation is T-transitive in order to make fuzzy inference, and if a relation is not T-Transitive it is possible to find another T-transitive fuzzy relation as close as possible with the initial fuzzy relation.

2 .2 Transitive closure

The T-transitive closure R^T of a fuzzy relation R is the lower relation that contains R and is T-transitive.

An algorithm used to compute the transitive closure is the following:

1) $R' = R \cup_{Max} (R o_{Sup\text{-}T} R)$
2) If $R' \neq R$ then $R := R'$ and go back to 1), otherwise stop and $R^T := R'$.

2 .3 A new T-transitivization algorithm

At '*On a new method to T-transitivize fuzzy relations'* [Garmendia & Salvador; 2000] it is proposed a new algorithm to T-transitivize fuzzy relations, obtaining a fuzzy T-transitive relation as close as possible from the initial fuzzy relation. If the initial relation is T-transitive then it is equal to the T-transitivized relation.

The transitivized relation keeps important properties as the μ-T-conditionality property and reflexivity that also preserves the transitive closure, but it also keeps some more properties as the invariance of the relation degree of every element with himself (or diagonal), and so it preserves α-reflexivity. The transitivity closure do not preserve α-reflexivity.

2 .4 Previous concepts

Let $E = \{a_1, ..., a_n\}$ be a finite set.

Definition 1*:* Let T be a triangular t-norm. A fuzzy relation $R: E \times E \to [0, 1]$ is **T-transitive** if $T(R(a,b), R(b,c)) \leq R(a,c)$ for all a, b, c in E.

Given a fuzzy relation R it is called element $a_{i,j}$ to the relation degree in [0, 1] between the elements a_i and a_j in E. So $a_{i,j} = R(a_i, a_j)$.

Definition 2: An element $a_{i,j}$ is called **T-transitive element** if $T(a_{i,k}, a_{k,j}) \leq a_{ij}$ for all k from 1 to n.

Algorithm: The proposed algorithm transform a fuzzy relation R^0 into another T-transitive relation R_T contained in R^0 in n^2-1 steps. In each step can be reduced some degrees so $R = R^0 \supseteq R^1 \supseteq ... \supseteq R^m \supseteq ... \supseteq R^{n^2-1} = R_T$.

The idea of this method is to get profit of the fact that each step makes sure that an element $a_{i,j}$ will be T-transitive for all further steps, and so it will be T-transitive in the final relation R_T. In summary, each step m+1 T-transitivize an element $a_{i,j}^m$ in R^m reducing other elements $a_{i,k}^m$ or $a_{k,j}^m$, when it is necessary, resulting that $a_{i,j}^r$ is T-transitive in R^r for all r≥m. To achieve this, it is important to choose in each step the minimum non T-transitivized element as the candidate to transitivize (reducing other elements). When choosing to transitivizate the minimum $a_{i,j}^m$ in R^m it is sure that $a_{i,j}^m = a_{i,j}^r$ for all r≥m (it will not change in further steps), because the reduction of other elements will not make it intransitive anymore and because $a_{i,j}^m$ is lower or equal further transitivized elements, it will not cause intransitivity and it will not be reduced.

Let τ be a set of pairs (i, j) where i, j are integers from 1 to n.

Definition 3: τ^m is a subset of τ defined by:

1) $\tau^0 = \varnothing$

2) $\tau^{m+1} = \tau^m \cup (i, j)$ if $a_{i,j}^m$ is the element in R^m chosen to be T-transitivized in the m+1 step.

So τ^m is the set of pairs (i, j) corresponding the T-transitivized elements in R^m and $(\tau^m)'$ is the set of n^2-m pairs (i, j) corresponding the not yet transitivized elements.

Building R^{m+1} from R^m: Let $a_{i,j}^m$ be the element in R^m that is going to be transitivized at step m+1 ($a_{i,j}^m = \text{Min}\{a_{v,w}^m$ such that $(v, w) \in (\tau^m)'\}$).

It is defined $a_{r,s}^{m+1}$ as

$$\begin{cases} J^T(a^m_{s,j}, a^m_{i,j}) & \text{if } r=i,\ T(a^m_{r,s}, a^m_{s,j}) > a^m_{i,j} \text{ and } a^m_{i,s} \leq a^m_{s,j} \\ J^T(a^m_{i,r}, a^m_{i,j}) & \text{if } s=j,\ T(a^m_{i,r}, a^m_{r,s}) > a^m_{i,j} \text{ and } a^m_{i,r} \geq a^m_{r,s} \\ a^m_{r,s} & \text{otherwise} \end{cases} \quad \textbf{(1)}$$

where J^T is the residual operator of the t-norm T, defined by $J^T(x, y) = \sup\{z / T(x, z) \leq y\}$.

If $T(a_{i,k}^m, a_{k,j}^m) > a_{ij}^m$ for some k, either $a_{i,k}^m$ or $a_{k,j}^m$ will reduce its degree (it could be chosen the minimum of both) to achieve that $T(a_{i,k}^{m+1}, a_{k,j}^{m+1}) \leq a_{ij}^{m+1} = a_{ij}^m$.

When choosing the minimum between $a_{i,k}^m$ and $a_{k,j}^m$ to reduce, if it is chosen the minimum one, the difference between R^m and R^{m+1} is lower, so if $a_{i,k}^m \leq a_{k,j}^m$ then $a_{i,k}^{m+1} = J^T(a_{k,j}^m, a_{ij}^m)$ and if $a_{i,k}^m > a_{k,j}^m$ then $a_{k,j}^{m+1} = J^T(a_{i,k}^m, a_{ij}^m)$. The degree of the rest of elements remains invariant ($a_{r,s}^{m+1} = a_{r,s}^m$).

3 The program

3.1 Program Description

It has been developed a program in C++ that generates a random fuzzy relation (shown at the top of the figure) and computes the Min-transitive closure, Prod-transitive closure and W-transitive closure (first row of relation in the figure), measuring the absolute value distance and euclidean distance with the initially generated fuzzy relation. It also computes the Min-transitivized relation, Prod-transitivized relation and W-transitivized relation (second row of relations in the figure), and also measures their distances with the same original fuzzy relation.

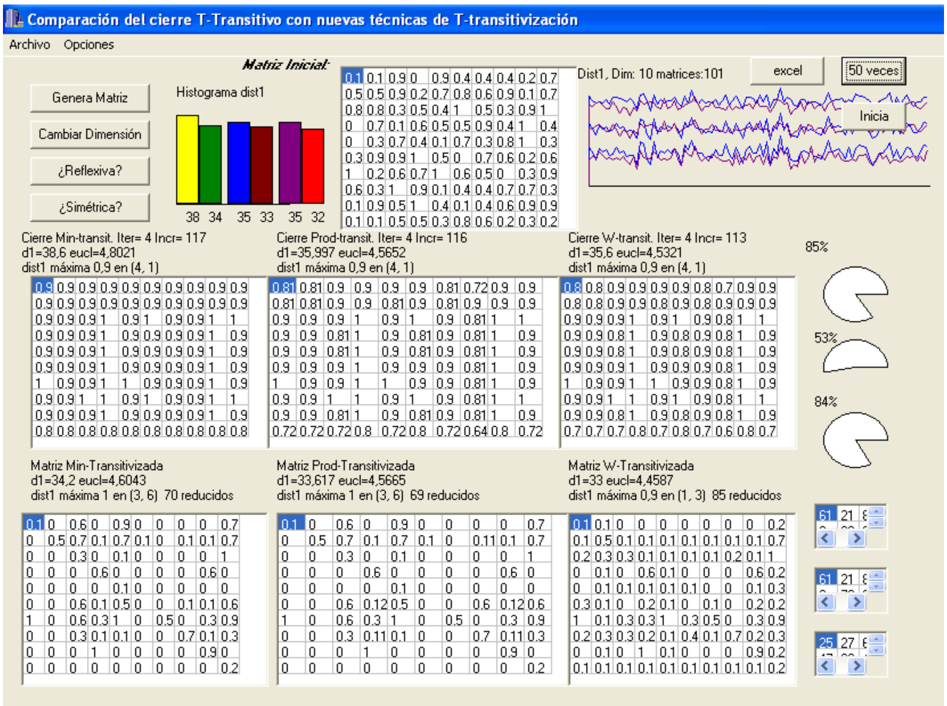

Fig. 1. General front-end of the program.

As an example, the program generates the following random fuzzy relation:

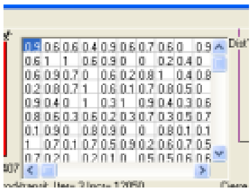

Fig. 2. Example of generated random fuzzy relation.

Computes the Min-transitive closure, Prod-transitive closure and W-transitive closure measuring the absolute value distance and euclidean distance with the initial fuzzy relation:

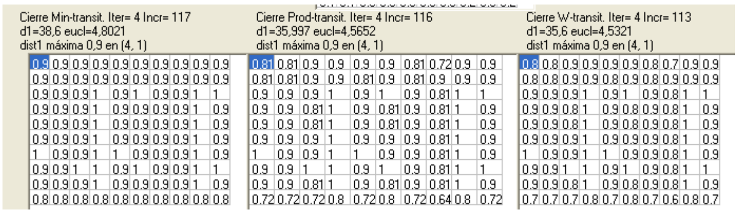

Fig. 3. Example of Min-Transitive closure, Prod-transitive closure and W-transitive closure of the random fuzzy relation of Fig. 2, measuring the absolute value distance and euclidean distance with the initial fuzzy relation.

It also computes the Min-transitivized relation, Prod-transitivized relation and W-transitivized relation (second row of relations in the figure), and also measures their distances with the same original fuzzy relation:

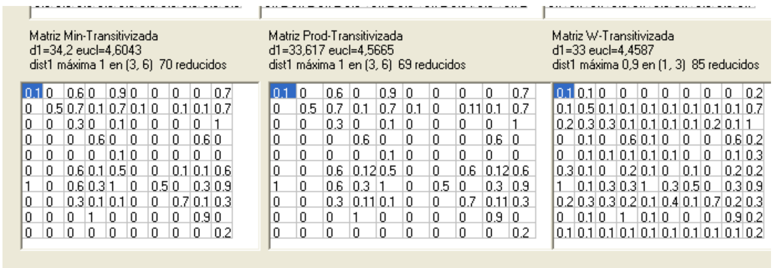

Fig. 4. Example of Min-transitivized relation, Prod-transitivized relation and W-transitivized relation of the random fuzzy relation of Fig. 2, measuring the absolute value distance and euclidean distance with the initial fuzzy relation

After doing this process 100 times, the program shows the percentage of times that the T-transitivized relation have a lower distance with the random relation than the distance of the T-transitive closure with the initial relation. For the minimum t-norm, the 85% of tries the distance with the Min-transitivized relation is lower than the distance with the Min-transitive closure. This percentage is 53% when T is the product t-norm, and for the Lukasiewicz t-norm a 84% of times is closer the W-transitivized relation than W-transitive closure:

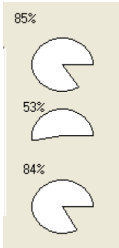

Fig. 5. After doing the Fig 2-3-4 process 101 times, the program shows the percentage of times that the T-transitivized relation have a lower distance with the random relation than the distance of the T-transitive closure with the initial relation.

It can also tell the program to generate reflexive fuzzy relation, and then there are generated two Min-preorders (the Min-transitive closure and the Min-transitivized relation), two Prod-preorders and two W-preorders.

When choosing to generate reflexive and symmetric random fuzzy relations their computed T-transitive closures will be generated T-indistinguishabilities. The original transitivization method described does not keep the symmetry but we already have developed a version to transitivize fuzzy relation keeping the symmetry (when reducing an element, it is also reduced its symmetric element) and then obtaining T-indistinguishabilities.

Fig. 6. Buttons to start a new process, and to choose the properties of the generated fuzzy relation, as the dimension, the reflexive property and the symmetric property.

Fig. 7. The program have buttons to repeat the process fifty times and keep the results in an Excel document

The histogram shows the absolute value distance of the last random generated fuzzy relation with the (in this order from the left to the right) Min-transitive closure, the Min-transitivized relation, the Prod-transitive closure, the Prod-transitivized relation, the W-transitive closure and the W-transitivized relation. The graph at the right of the picture compares the absolute value distances of both T-transitivization methods for the t-norms (in this order, from the upper to the lower graphs) minimum, product and Lukasiewicz for the last hundred of random fuzzy relations. In most cases, the distances of the T-transitivized relation is lower than the distances with the T-transitive closure for the three t-norms.

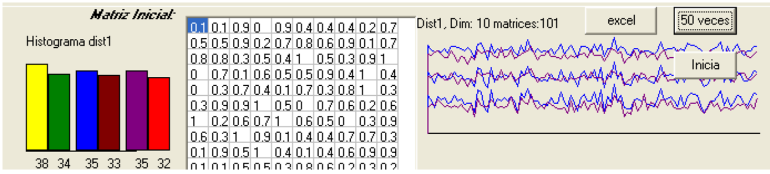

Fig. 8. The histogram shows the absolute value distance of the last random generated fuzzy relation with the Min-transitive closure, the Min-transitivized relation, the Prod-transitive closure, the Prod-transitivized relation, the W-transitive closure and the W-transitivized relation. The graph at the right of the picture compares the absolute value distances of both T-transitivization methods for the t-norms minimum, product and Lukasiewicz for the last hundred of random fuzzy relations.

The program has been scheduled to generate one hundred of random fuzzy relations for each dimension from two to one hundred. The average distances for each dimension have been saved in an Excel document.

4 Program work

It has been run the program one hundred times for each dimension from two to one hundred, it is, the program has generated 9900 random fuzzy relations, computing their T-transitive closures and their T-transitivized relations for different t-norms, and computing their average distance of absolute value and euclidean for each dimension.

The function in the graph below represents, for each dimension, the average absolute value distance with their W-transitive closure (the line of higher distances) and the W-transitivized relation. The aspect of the results could change when using other distances, but it is got the same looking for the three t-norms used.

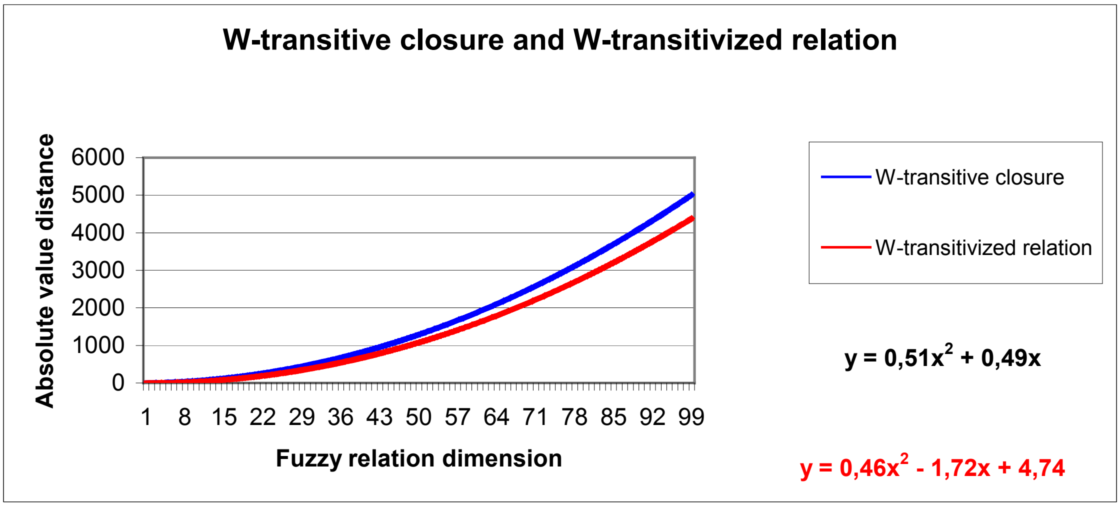

Fig. 9. Average of the absolute value distances of 100 random relations with their W-transitive closure and W-transitivized relation for each dimension from two to one hundred.

The functions for those average distances for the t-norm minimum, product and Lukasiewicz are the following:

Table 1. Interpolation function of the average absolute value distance of the W-transitive closure and W-transitivized relation of one hundred random fuzzy relations for each dimension from two to one hundred.

Absolute value distance	**Min**	**Prod**	**W**
Transitive Closure	$y=0,5x^2+1,19x-16,27$	$y=0,6x^2-3,4x+5$	$y=0,51x^2+0,49x$
Transitivized relation	$y=0,47x^2-1,27x+5,9$	$y=0,47x^2-1,23x+5,1$	$y=0,46x^2-1,72x+4,74$

The average distances of the generated relations with the transitive closure is higher that for the transitivized relation for all dimensions and for all t-norms.

However when using the euclidean distances it is also got higher distances for the T-transitive closure for the three t-norms, but we get linear functions:

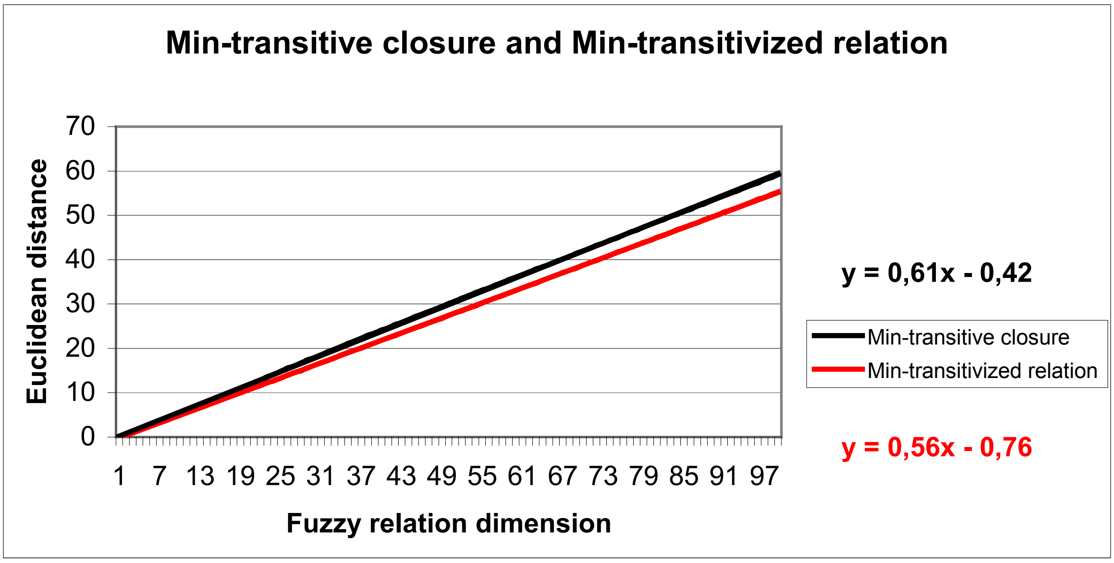

Fig. 10. Average of the euclidean distances of the Min-transitive closure and Min-transitivized relation of one hundred random fuzzy relations for each dimension from two to one hundred.

The linear functions resulting when using euclidean distances are the following:

Table 2: Interpolation function of the average euclidean distance of the T-transitive closure and T-transitivized relation of one hundred random fuzzy relations for each dimension from two to one hundred.

Euclidean distances	Min	Prod	W
Transitive Closure	y=0,61x-0,42	y=0,61x-0,63	y=0,61x-0,68
Transitivized relation	y=0,56x-0,76	y=0,56x-0,77	y=0,56x-1,19

As the mean distances of the T-transitive closure are higher than the mean distances for the T-transitivized relations, we have study the difference. The graph below shows those difference between the means using the absolute value distance and the minimum t-norm, for dimensions from two to one hundred:

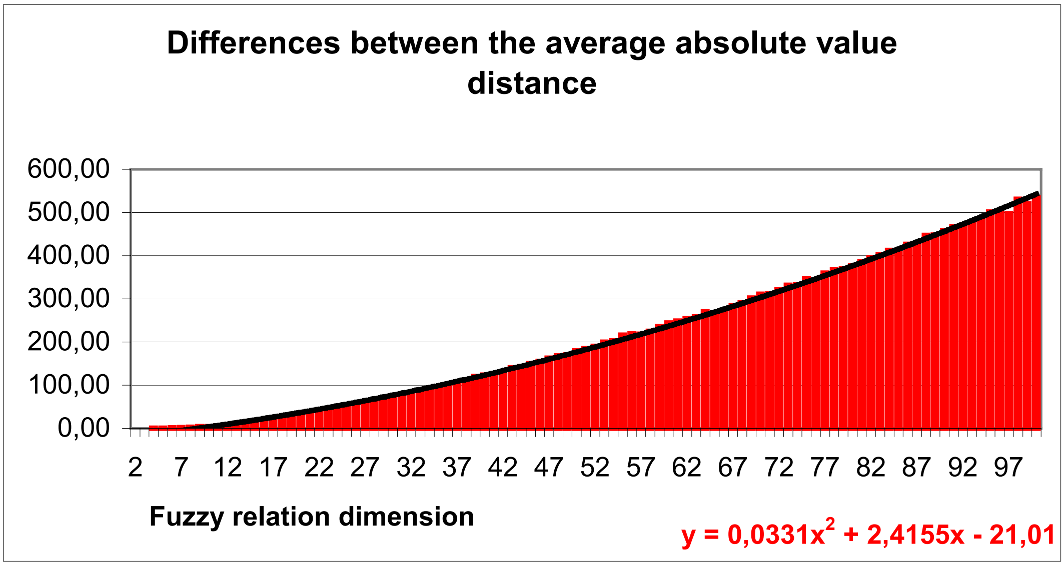

Fig. 11. Differences between the average absolute value distance of the 100 generated relations with their Min-transitive closure and Min-transitivized relations, for dimensions from 2 to 100.

Some statistical values for those 9900 generated relations and their transitivized relations using the absolute value distance are the following:

Table 3: Statistical values of all absolute value distances with the transitivized relations for the 9900 fuzzy relations generated.

Absolute value distance	Minimum		Product		Lukasiewicz	
	Transitive closure	Algorithm	Transitive closure	Algorithm	Transitive closure	Algorithm
Mean (average)	1702	1492	1701	1491	1701	1456
Standard deviation	1515,47	1350,05	1516,42	1349,09	1516,60	1326,66
Second quartile	1296,3	1111,0	1296,3	1109,7	1296,3	1079,1
First quartile	335,4	279,0	334,3	278,5	334,2	264,0
Third quartile	2846,8	2504,9	2846,8	2502,7	2846,8	2448,8

Table 4: Statistical values of all Euclidean distances with the transitivized relations for the 9900 fuzzy relations generated.

Euclidean distance	Minimum		Product		Lukasiewicz	
	Transitive closure	Algorithm	Transitive closure	Algorithm	Transitive closure	Algorithm

Mean (average)	30	27	30	27	30	27
Standard deviation	17,38	16,22	17,47	16,22	17,49	16,16
Second quartile	30,1	27,4	30,1	27,4	30,1	26,9
First quartile	15,2	13,6	15,1	13,6	15,1	13,1
Third quartile	44,6	41,3	44,6	41,3	44,6	40,7

5 Results analysis

After generating 100 random fuzzy relations for all dimensions from 2 to 100, and compute their average distance with the T-transitive closure and with the T-transitivized relation, we have seen for any distance, for any t-norm and for any dimension that the T-transitivized relation is closer to the initial relations than the T-transitive closure.

When obtaining global measures for the 9900 relations, the transitivized relation is also closer than the transitive closure, and has lower dispersion.

6 Conclusions

The T-transitivization algorithm gives closer T-transitive relations than the T-transitive closure for any dimension and any t-norm. They are also different, because gives T-transitive relations contained in the initial relation.

The T-transitive closure is uniquely defined, however we can find several T-transitive relations contained in the initial relation.

It is proven [Garmendia & Salvador; 2000] that the T-transitivization algorithm keeps the reflexivity and α-reflexivity. However the T-transitive closure keeps reflexivity, but not α-reflexivity. However the algorithm does not keep symmetry as the transitive closure does, and so it does not produce T-indistinguishabilities from reflexive and symmetric relations. We have already developed a new version that does keep it, reducing the symmetric element of all reduced elements.

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