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Classifying Fuzzy Measures

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Abstract. This paper presents a state of art on the latest concepts of measure, from the additive measures, to monotone fuzzy measures and the latest monotone measures in relation to a preorder that gives an ordering for a measurable characteristic.

1 Introduction

The discovery of useful information is the essence of any data mining process. Decisions are not usually taken based on complete real world data, but most of the times they deal with uncertainty or lack of information. Therefore the real world reasoning is almost always approximate. However it is not only necessary to learn new information in any data mining process, but it is also important to understand why and how the information is discovered. Most data mining commercial products are black boxes that do not explain the reasons and methods that have been used to get new information. However the 'why and how' the information is obtained can be as important as the information on its own. When approximate reasoning is done, measures on fuzzy sets and fuzzy relations can be proposed to provide a lot of information that helps to understand the conclusions of fuzzy inference processes. Those measures can even help to make decisions that allow to use the most proper methods, logics, operators for connectives and implications, in every approximate reasoning environment.

The latest concepts of measures in approximate reasoning is discussed and a few measures on fuzzy sets and fuzzy relations are proposed to be used to understand why the reasoning is working and to make decisions about labels, connectives or implications, and so a few useful measures can help to have the best performance in approximate reasoning and decision making processes.

Before some measures on fuzzy sets and fuzzy relations are proposed, this chapter collects all the latest new concepts and definitions on measures, and shows a few graphics that make a clear picture on how those measures can be classified.

Some important measures on fuzzy sets are the entropy measures and specificity measures. The entropy measures give a degree of fuzziness of a fuzzy set, which can be computed by the premises or outputs of an inference to know an amount of uncertainty crispness in the process. Specificity measures of fuzzy sets give a degree of the utility of information contained in a fuzzy set.
Other important measures can be computed on fuzzy relations. For example, some methods to measure a degree of generalisation of the MODUS PONENS property in fuzzy inference processes are proposed.

2 The Concept of Measure

The concept of measure is one of the most important concepts in mathematics, as well as the concept of integral respect to a given measure. The classical measures are supposed to hold the additive property. Additivity can be very effective and convenient in some applications, but can also be somewhat inadequate in many reasoning environ ments of the real world as in approximate reasoning, fuzzy logic, artificial intelligence, game theory, decision making, psychology, economy, data mining, etc., that require the definition of non additive measures and a large amount of open problems. For example, the efficiency of a set of workers is being measured, the efficiency of the same people doing teamwork is not the addition of the efficiency of each individual working on their own.

The concept of fuzzy measure does not require additivity, but it requires monotonicity related to the inclusion of sets. The concept of fuzzy measure can also be generalised by new concepts of measure that pretend to measure a characteristic not really related with the inclusion of sets. However those new measures can show that "x has a higher degree of a particular quality than y" when x and y are ordered by a preorder (not necessarily the set inclusion preorder).

The term fuzzy integral uses the concept of fuzzy measure. There are some important fuzzy integrals, as C*hoquet* integral in 1974, which does not require an additive measure (as Lebesgue integral does). Michio Sugeno gives other new integral in 1974 for fuzzy sets, and so does *David Schmeidler* in 1982 for decision theory.

2.1 Preliminaries

A measurable space is a couple $(X, \wp)$ where X is a set and $\wp$ is a **σ-algebra** or set of subsets of X such that:

1. $X \in \wp$.
2. Let A be a subset of X. If $A \in \wp$ then $A' \in \wp$.
3. If $A_n \in \wp$ then $\bigcup_{n=1}^{\infty} A_n \in \wp$.

For example, when X is the set of real numbers and $\wp$ is the σ-algebra that contains the open subsets of X, then $\wp$ is the well-known Borel σ-algebra.

Note:

The classical concept of measure considers that $\wp \subseteq \{0, 1\}^X$, but this consideration can be extended to a set of fuzzy subsets $\Im$ of X, $\Im \subseteq [0, 1]^X$, satisfying the properties of measurable space ($[0, 1]^X$, $\Im$).

2.2 Definition of Additive Measure:

Let (X, $\wp$) be a measurable space. A function m: $\wp \to [0, \propto)$ is an σ-additive measure when the following properties are satisfied:

1. $m(\emptyset) = 0$
2. If A_n, n = 1, 2, ... is a set of disjoint subsets of $\wp$ then

$$m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$$

The second property is called **σ-additivity,** and the **additive** property of a measurable space requires the σ-additivity in a finite set of subsets A_n.

A well-known example of **σ-additive** is the probabilistic space (X, $\wp$, p) where the probability p is an additive measure such that p(X)=1 and p(A)=1–p(A') for all subsets $A \in \wp$.

Other known examples of σ-additive measure are the Lebesgue measures defined in 1900 that are an important base of the XX century mathematics.

2.3 Definition of Normal Measure

Let (X, $\wp$) be a measurable space. A measure m: $\wp \to [0, 1]$ is a normal measure if there exists a minimal set A_0 and a maximal set A_m in $\wp$ such that:

1. $m(A_0) = 0$
2. $m(A_m) = 1$

For example, the measures of probability on a space (X, $\wp$) are normal measures with $A_0=\emptyset$ and $A_m=X$. The Lebesgue measures are not necessarily normal.

2.4 Definition of Sugeno Fuzzy Measure [17]

Let $\wp$ be an σ-algebra on a universe X. A **Sugeno fuzzy measure** is g: $\wp \to [0, 1]$ verifying:

1. $g(\emptyset) = 0$, $g(X) = 1$
2. If A, $B \in \wp$ and $A \subseteq B$ then $g(A) \leq g(B)$
3. If $A_n \in \wp$ and $A_1 \subseteq A_2 \subseteq ...$ then $\lim_{n\to\infty} g(A_n) = g(\lim_{n\to\infty} A_n)$

Property 2 is called monotony and property 3 is called Sugeno´s convergence.

The Sugeno measures are monotone but its main characteristic is that additivity is not needed.

Probability, credibility and plausibility measures are Sugeno measures. The possibility measures on possibility distributions are Sugeno measures.

2.5 Theory of Evidence

The theory of evidence is based on two dual non-additive measures: belief measures and plausibility measures.

Given a measurable space (X, $\wp$), a belief measure is a function Bel: $\wp \to [0, 1]$ verifying the following properties:

1. $Bel(\varnothing) = 0$
2. $Bel(X) = 1$
3. $Bel(A \cup B) \geq Bel(A) + Bel(B)$

Property 3 is called **superadditivity**. When X is infinite, the superior continuity of the function Bel is required. For every $A \in \wp$, Bel(A) is interpreted as a belief degree for some element to be in the set A.

From the definition of belief measure, it can be proved that $Bel(A) + Bel(A') \leq 1$.

Given a belief measure, its dual plausibility measure can be defined as $Pl(A) = 1 - Cred(A')$.

Given a measurable space (X, $\wp$) a measure of plausibility is a function Pl: $\wp \to [0, 1]$ such that

1. $Pl(\varnothing) = 0$.
2. $Pl(X) = 1$.
3. $Pl(A \cup B) \leq Pl(A) + Pl(B)$.

Property 3 is called **subadditivity**.

When X is infinite, the inferior continuity of the function Pl is required.

It can be proved that $Pl(A) + Pl(A') \geq 1$.

The measures of credibility and plausibility are defined by a function m: $\wp \to [0, 1]$ such that $m(\varnothing) = 0$ and $\sum_{A \in \wp} m(A) = 1$ where m represents a proportion of the shown evidence that an element of X is in a subset A.

2.6 Theory of Possibility

The theory of possibility is a branch of theory of evidence where the plausibility measures verify that $Pl(A \cup B) = \max\{Pl(A), Pl(B)\}$. Such plausibility measures are called **possibility measures**. In the theory of possibility, the belief measures satisfy that $Bel(A \cap B) = \min\{Bel(A), Bel(B)\}$ and are called **necessity measures**.

Definition 1 [14]
Let (X, $\wp$) be a measurable space. A possibility measure is a function Π: $\wp \to [0, 1]$ that verifies the following properties:

1. $\Pi(\varnothing) = 0$, $\Pi(X) = 1$.
2. $A \subseteq B \Rightarrow \Pi(A) \leq \Pi(B)$

3. $\Pi(\bigcup_{i\in I} A_i) = \sup_{i\in I} \{\Pi(A_i)\}$ for a set of indexes I.

The possibility measures are sub additive normal measures.

Definition 2 [14]

Let (X, $\wp$) be a measurable space. A necessity measure is a function Nec: $\wp \to [0, 1]$] that verifies the following properties:

1. $Nec(\emptyset) = 0$, $Nec(X) = 1$.
2. $A \subseteq B \Rightarrow Nec(A) \le Nec(B)$
3. $Nec(\bigcap_{i\in I} A_i) = \inf_{i\in I} \{Nec(A_i)\}$ for any set I.

Possibility measures are plausibility measures and necessity measures are belief measures, so:

1. $\Pi(A) + \Pi(A') \ge 1$.
2. $Nec(A) + Nec(A') \le 1$.
3. $Nec(A) = 1 - \Pi(A')$.
4. $\max\{\Pi(A), \Pi(A')\} = 1$.
5. $\min\{Nec(A), Nec(A')\} = 0$.
6. $Nec(A) > 0 \Rightarrow \Pi(A) = 1$.
7. $\Pi(A) < 1 \Rightarrow Nec(A) = 0$.

The *Shafer* [26] theory of evidence stands that the probability of an element or a set is related to its complementary one. It includes concepts of 'low probability' and 'high probability', that are related to the measures of possibility and necessity in the sense that for any subset A, $Nec(A) \le P(A) \le \Pi(A)$.

The theory of possibility also stands on fuzzy sets, where $\wp$ is a family of fuzzy subsets in X.

A measure of possibility is not always a Sugeno fuzzy measure [22]. However a normal possibility distribution on a finite universe X is a Sugeno measure.

2.7 Definition of Fuzzy Measure

Let (X, $\wp$) be a measurable space. A function m: $\wp \to [0, \infty)$ is a fuzzy measure (or monotone measure) if it verifies the following properties:

1. $m(\emptyset) = 0$.
2. If $A, B \in \wp$ and $A \subseteq B$ then $m(A) \le m(B)$.

Property 2 is called monotony.

For example, all σ-additive measures (as probability) are fuzzy measures. Some other fuzzy measures are the necessity measures, the possibility measures and the Sugeno measures.

2.8 Definition of Fuzzy Sugeno λ-Measure

Sugeno [27] introduces the concept of fuzzy λ-measure as a normal measure that is λ-additive. So the fuzzy λ-measures are fuzzy (monotone) measures.

Let $\lambda \in (-1, \infty)$ and let $(X, \wp)$ be a measurable space. A function $g_\lambda: \wp \to [0, 1]$ is a fuzzy λ-measure if for all disjoint subsets A , B in $\wp$, $g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A)\, g_\lambda(B)$.

For example, if $\lambda = 0$ then the fuzzy λ-measure is an additive measure.

2.9 S-Decomposable Measures

The S-decomposable measures provide a general concept of the fuzzy λ-measures and the possibility measures.

Let S be a t-conorm, and let $(X, \wp)$ be a measurable space. A S-decomposable measure is a function $m: \wp \to [0, 1]$ that verifies the following conditions.

1. $m(\emptyset) = 0$.
2. $m(X) = 1$.
3. For all disjoint subsets A and B in $\wp$, $m(A \cup B) = S(m(A), m(B))$.

The property 3 is called ***S-additivity.***

For example, the probability measures are W*-decomposable measures, where W* is the Łukasiewicz t-conorm. The W^*_λ-decomposable measures, where W^*_λ is the t-conorm $W^*_\lambda(x, y) = x + y + \lambda xy$ are fuzzy λ-measures.

Let m be a S-decomposable measure on $(X, \wp)$. If X is finite then given a subset A in $\wp$, $m(A) = \underset{x \in A}{S} \{m(\{x\})\}$.

2.10 Fuzzy $\prec$ -Measure (Fuzzy Preorder-Monotone Measure)

Trillas and Alsina [33] give a general definition of fuzzy measure. When a characteristic, namely –volume, weight, etc.– needs to be measured on the elements of a set X, a preorder relation that allows to stand that "*x* shows the characteristic less than y shows it" for all x and y in X is necessary to be set. That reflexive and transitive relation is denoted $x \prec y$.

A fuzzy $\prec$ -measure is defined as follows:

Let $\prec$ be a preorder, for which 0 is a minimal element in X and **1** is a maximal element in X. Then a fuzzy $\prec$-measure is a function $m: \wp(X) \to [0, 1]$ that verifies the following conditions:

1. m(0)=0
2. m(1)=1
3. If $x \prec y$ then $m(x) \le m(y)$.

A good example of fuzzy $\prec$ -measure on the set of natural numbers N is the Sarkovskii measure, which is defined as a measure of approximately even numbers, given by the following function m:

$$m(n) = \begin{cases} 1 & \text{if } n = 2^k \text{ for } k = 0,1,2,\ldots \\ 0 & \text{if } n = 2^k + 1 \text{ for } k = 1,2,\ldots \\ 1 - \dfrac{1}{2^k} & \text{if } n = 2^k(2p+1) \text{ for } k = 1,2,\ \ldots\ p = 1,2,\ldots \end{cases}$$

Then *m* is a fuzzy $\prec$-measures, not for the normal natural numbers order, but for the *Sarkovskii* order, for which the lowest number is 3, and the greatest number is 1. It is a well-known order used in dynamic systems and given defined as follows:

$3 \prec 5 \prec 7 \prec \ldots \prec 2.3 \prec 2.5 \prec \ldots \prec 2^2.3 \prec 2^2.5 \prec \ldots \prec 2^3.3 \prec 2^3.5 \prec \ldots 2^3.3 \prec 2^3 \prec 2^2 \prec 2 \prec 1$

Other fuzzy $\prec$-measure are all previous defined fuzzy measures, which are monotone measures with respect to the set inclusion preorder, that is now generalised in both classic set inclusion and fuzzy set inclusion cases.

The *Sugeno* [27] fuzzy measure concept is also generalised: if $\wp$ is a partial order lattice, then $x \prec y$ if and only if $x \wedge y = x$, and the three Sugeno properties are satisfied. If the lattice is ortocomplemented then there exists a dual function $m^*(x)=1-m(x')$ that also is a fuzzy $\prec$-measure.
Then, the probability measure on a Boole algebra of probabilistic successes is also a fuzzy $\prec$-measure.

Let $\wp$ be the set of fuzzy subsets on a given set, the entropy measure introduced by De Luca and Termini, and the possibility or necessity measures [14] are also a fuzzy $\prec$-measures.

3 Some Measures On Fuzzy Sets and Fuzzy Relations

3.1 Entropy or Measures of Fuzziness

Let X be a set and let $\wp(X)$ be the set of fuzzy sets on X. The measures of fuzziness or **entropies** give a degree of fuzziness for every fuzzy set in $\wp$.
Some entropy measures have influence from the *Shannon* probabilistic entropy, which is commonly used as measures of information.

De Luca and Termini consider the entropy E of a fuzzy set of $\wp(X)$ as a measure that gives a value in $[0, \propto]$ and satisfies the following properties:

1. E(A) = 0 if A is a crisp set.
2. E(A) is maximal if A is the constant fuzzy set $A(x) = \dfrac{1}{2}$ for all $x \in X$.
3. $E(A) \geq E(B)$ if A is 'more fuzzy' than B by the 'sharpen' order.
4. E(A) = E(A').

Note that the defined entropy measure of a fuzzy set is a fuzzy $\prec$-measure where the $\prec$ preorder is the $\leq_S$ sharpen order, in which $B \leq_S A$ if for any element x in the universe of discourse when $A(x) \leq \frac{1}{2}$ then $B(x) \leq A(x)$ and when $A(x) \geq \frac{1}{2}$ then $B(x) \geq A(x)$

Kaufmann proposes a fuzziness index as a normal distance. Other authors as Yager [41] and *Higashi* and *Klir* [14] understand the entropy measures as the difference between a fuzzy set and its complementary fuzzy set.

3.2 Measures of Specificity

The specificity measures introduced by Yager [47] are useful as measures of tranquillity when making a decision. Yager introduces the specificity-correctness trade-off principle. The output information of expert systems and other knowledge-based systems should be both specific and correct to be useful. Yager suggests the use of specificity in default reasoning, in possibility-qualified statements and data mining processes, giving several possible manifestations of this measure. Kacprzyk describes its use in a system for inductive learning. Dubois and Prade [5] introduce the minimal specificity principle and show the role of specificity in the theory of approximate reasoning. Higashi and Klir [14] introduce a closely related idea called non-specificity. The concept of granularity introduced by Zadeh [53] is correlated with the concept of specificity.

Let X be a set with elements $\{x_i\}$ and let $[0, 1]^X$ be the class of fuzzy sets of X. A measure of specificity Sp is a function Sp: $[0, 1]^X \to [0, 1]$ such that:

1. $Sp(\mu) = 1$ if and only if μ is a singleton ($\mu = \{x_1\}$).
2. $Sp(\emptyset) = 0$
3. If μ and η are normal fuzzy sets in X and $\mu \subset \eta$, then $Sp(\mu) \geq Sp(\eta)$.

A general expression [8] that can be used to build measures of specificity from three t-norms and negations is an application Sp_T: $[0, 1]^X \to [0, 1]$ defined by $\mathbf{Sp_T(\mu) = T_1(a_1, N(T_2{}^*_{j=2,\dots,n}\{T_3(a_j, w_j)\})}$ where μ is a fuzzy set in a finite set X, and a_i is the membership degree of the element x_i ($\mu(x_i)=a_i$), the membership degrees $a_i \in [0, 1]$ are totally ordered with $a_1 \geq a_2 \geq \dots \geq a_n$, N is a negation, let T_1 and T_3 be any t-norms, $T_2{}^*$ a n-argument t-conorm and $\{w_j\}$ is a weighting vector.

For example, when N is the negation $N(x) = 1-x$, T_1 and T_2 are the Łukasiewicz t-norm defined by $T_1(a, b) = \max\{0, a+b-1\}$, so $T_2{}^*(a_1, \dots, a_n) = \min\{1, a_1+ \dots +a_n\}$, and T_3 is the product, then the previous expression gives Yager´s [47] linear measure of specificity, defined as

$$Sp(\mu) = a_1 - \sum_{j=2}^{n} w_j a_j.$$

The measures of specificity are not monotone measures, because the measure of specificity of a fuzzy set is lower when some membership degrees that are not the highest degree are increased. However the measures of specificity of fuzzy sets are fuzzy $\prec$-measures, where $\prec$ is a preorder that classifies the fuzzy sets by the utility of the contained information, or by a given distance to a singleton.

3.3 Measures of μ-T-Unconditionality

The μ-T-conditionality property of fuzzy relations generalises the modus ponens property when making fuzzy inference. A fuzzy relation R: $E_1 \times E_2 \rightarrow [0,1]$ is μ-T-conditional if and only if $T(\mu(a), R(a, b)) \leq \mu(b)$ for all (a, b) in $E_1 \times E_2$.
Some ways to measure a degree of verification of this property are discussed, which are monotonous measures on the measurable space $(\mathfrak{R}, \mathfrak{I}, M)$, where $\mathfrak{R}$ is the set of fuzzy relations R: $E_1 \times E_2 \rightarrow [0, 1]$, $\mathfrak{I}$ the set of measurable subsets of $\mathfrak{R}$ and M is a measure of μ-T- unconditionality. There are two ways to define those measures [9]. A first way computes a generalised distance between a fuzzy relation R and the greatest μ-T-conditional relation that is contained in R. The other way measures the difference between $T(\mu(a), R(a, b))$ and $\mu(b)$ in all points (a, b) in which R is not μ-T-conditional.

The measures of **μ-T- unconditionality of fuzzy relations are monotone measures on the measurable space** $(\mathfrak{R}, \mathfrak{I}, M)$ where $\mathfrak{R}$ is the set of fuzzy relations R: $E_1 \times E_2 \rightarrow [0, 1]$, $\mathfrak{I}$ is the set of measurable subsets of $\mathfrak{R}$ and M is a measure of μ-T-unconditionality.

4 Conclusions

The latest concepts of fuzzy measure are presented.
Some measures are relevant to understand the process of inference, even when these are neither additive nor monotone. Proposals for non-monotone measures on fuzzy sets (entropy and specificity) are mentioned and are classified.
Some of the main measures to understand the information on the premises or conclusions in approximate reasoning are presented and classified in the context of the last concepts of fuzzy measures.

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