How to Make T-Transitive a Proximity Relation

L. Garmendia and J. Recasens

***Abstract*—Three ways to approximate a proximity relation R (i.e., a reflexive and symmetric fuzzy relation) by a T-transitive one where T is a continuous Archimedean t-norm are given. The first one aggregates the transitive closure $\overline{R}$ of R with a (maximal) T-transitive relation B contained in R. The second one computes the closest homotecy of $\overline{R}$ or B to better fit their entries with the ones of R. The third method uses nonlinear programming techniques to obtain the best approximation with respect to the Euclidean distance for T the Łukasiewicz or the product t-norm. The previous methods do not apply for the minimum t-norm. An algorithm to approximate a given proximity relation by a Min-transitive relation (a similarity) is given in the last section of the paper.**

***Index Terms*—Aggregation operator, proximity, quasi-arithmetic mean, representation theorem, similarity, T-indistinguishability operator, tolerance relation, transitive closure, transitive opening.**

I. Introduction

A PROXIMITY matrix or relation on a finite universe X is a reflexive and symmetric fuzzy relation R on X. In many applications for coherence imposition or knowledge learning reasons, transitivity of R with respect to a t-norm T is required. T-transitive approximation methods for proximities are specially useful in many artificial intelligence areas, such as fuzzy clustering [5], nonmonotonous reasoning [12], fuzzy database modeling [10], [14], decision making, and approximate reasoning [2], [15] applications. In these cases, R must be replaced by a new relation E also satisfying transitivity; such relations are called T-indistinguishability operators. Of course, it is desirable that E is as close as possible to R. This paper presents three ways to find close transitive relations to R in a reasonable way—i.e., easy and rapid to generate—when the t-norm is continuous Archimedean and a fourth one for the minimum t-norm.

There are, of course, several ways to calculate the closeness of two fuzzy relations, many of them related to some metric. In this paper, we propose a way related to the natural indistinguishability operator E_T associated to T, so that the degree of closeness or similarity between two fuzzy relations R and S is calculated aggregating the similarity of their respective entries using the quasi-arithmetic mean generated by an additive generator of T.

Also, the Euclidean metric will be used as an alternative method to compare fuzzy relations.

Trying to find the closest E to R can be very expensive. Indeed, if n is the cardinality of the universe X, the transitivity of T-indistinguishability operators can be modeled by $3\binom{n}{3}$ inequalities and they lay in the region of the $\binom{n}{2}$-dimensional space defined by them. The calculation of E becomes then a nonlinear programming problem. For the t-norm of Łukasiewicz and using the Euclidean distance to compare fuzzy relations, the problem is a classical quadratic nonlinear programming one and standard methods to solve it can be applied. Also, for the product t-norm standard, nonlinear programming algorithms can be used [3]. For other Archimedean t-norms or distances to measure the similarity between fuzzy relations simpler methods to find a close E to R are needed.

Usually, the proximity relation R is approximated by its transitive closure, by one of their transitive openings (i.e., maximal T-indistinguishability operators B among the set of T-indistinguishability operators smaller than or equal to R) or using the representation theorem for T-indistinguishability operators obtaining in this case a T-indistinguishability operator $\underline{R}$ smaller than or equal to R. The transitive closure has all its values greater than or equal to the corresponding values of R while, the transitive openings and $\underline{R}$ have their values below the corresponding ones of R. Methods to find T-indistinguishability operators with some values greater and some values smaller than the ones of R will generate better approximations of R.

It appears reasonable to aggregate $\overline{R}$ and B or $\underline{R}$ to obtain a new T-indistinguishability operator closer to R than $\overline{R}$, B, or $\underline{R}$. This idea will be developed in Section III.

If E is a T-indistinguishability operator, then the powers $E^{(p)}$ $p > 0$ of E with respect to the t-norm T are T-indistinguishability operators as well (see the next section for the definition of $E^{(p)}$). This allows us to increase or decrease the values of E, since $E^{(p)} \leq E^{(q)}$ for $p \geq q$. So, we can decrease the values of the transitive closure or increase the ones of an operator smaller than R to find better approximations of it. Section IV is devoted to this idea.

In Section V, nonlinear programming techniques are applied to find the closest T-indistinguishability operator to a given proximity with respect to the Euclidean distance.

The methods used for Archimedean t-norms cannot be applied for the minimum t-norm. Section VI provides an easy algorithm to compute better approximations of a fuzzy relation by a Min-transitive relation (a similarity [17]) than its transitive closure or their transitive openings.

Manuscript received March 4, 2008; revised July 22, 2008. First published August 4, 2008; current version published February 4, 2009. This work was partially supported by the Dirección General de Investigación Ciencia y Técnica (DGICYT) under Projects TIN2006-14311 and TIN2006-06190.

L. Garmendia is with the Department of Ingeniería del Software e Inteligencia Artificial, Facultad de Informática, Universidad Complutense de Madrid, 28040 Madrid, Spain (e-mail: lgarmend@fdi.ucm.es).

J. Recasens is with the Department of Matemàtiques i Informàtica, ETS Arquitectura del Vallès, Universitat Politècnica de Catalunya, 08190 Sant Cugat del Vallès, Spain (e-mail: j.recasens@upc.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TFUZZ.2008.2003128