Soft Computing ■■ (2003) ■■■–■■■
DOI 10.1007/s00500-003-0341-8

ORIGINAL PAPER

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On measuring μ-T-inconditionality of fuzzy relations

Published online: ■ ■ ■

Abstract In this paper two methods are proposed to measure the μ-T-inconditionality character of any fuzzy relation for any continuous t-norm T, and it is studied when both methods result to be equivalent.

Keywords: Fuzzy relation · Measures of conditionality · Conditionalized fuzzy relation

Introduction

The fuzzy relations that are used to make fuzzy inference should generalise the Modus Ponens property. A way to do this is through the μ-T-conditionality property [1–3] of fuzzy relations, a worldwide known generalised modus ponens definition.

In some environments it have been used fuzzy relations that not always verify the μ-T-conditionality property. In this case it looks interesting to study a degree of satisfaction of this property to find measures for the μ-T-conditionality property of fuzzy relations.
The μ-T-inconditionality of a fuzzy relation refers to the subset of the domain of a fuzzy relation in which it is not a μ-T-conditional.

In this paper two ways for measuring the μ-T-inconditionality property of fuzzy relations are proposed. A first way computes a generalized distance between a fuzzy relation R and the greatest μ-T-conditional relation that is contained in R. The other way measures the difference between $T(\mu(a), R(a, b))$ and μ(b) in all points (a, b) in which R is not μ-T-conditional.

It is proven that when T is any continuous t-norm, and when a generalized distance defined from a residuated operator of the T-norm is used, both methods give the same measures of μ-T-inconditionality of fuzzy relations.

Preliminaries

1. Let E_1, E_2 be two sets, let E be the set $E_1 \cup E_2$, let μ: $E \to [0,1]$ be a fuzzy set and let T be a continuous t-norm. A fuzzy relation R: $E_1 \times E_2 \to [0,1]$ is μ-T-conditional if and only if $T(\mu(a), R(a,b)) \leq \mu(b)$ for all (a, b) in $E_1 \times E_2$.
2. The μ-T-inconditionality region of a fuzzy relation, INC_T^μ (R), is defined as the subset of $E_1 \times E_2$ in which R is not μ-T-conditional, that is:

$$\mathrm{INC}_T^\mu(R) = \{(a,b) \in E_1 \times E_2 | T(\mu(a), R(a,b)) > \mu(b)\}$$

3. Let T_R^μ be the fuzzy relation defined by $T_R^\mu(a,b) = T(\mu(a), R(a,b))$. For example, Min_R^μ is the relation defined by $\mathrm{Min}_R^\mu(a,b) = \mathrm{Min}(\mu(a), R(a,b))$. Then the μ-T-inconditionality region of a fuzzy relation may also be expressed as:

$$\mathrm{INC}_T^\mu(R) = \{(a,b) \in E_1 \times E_2 | T_R^\mu(a,b) > \mu(b)\}$$

4. From the well known operation in [0, 1], J^T(x, y) = Sup z: T(x, z) $\leq$ y, [5–7], let J_μ^T be defined as the residual relation $J_\mu^T(a,b) = J^T(\mu(a), \mu(b))$. This allows a third way of expressing the μ-T-inconditionality region of a fuzzy relation:

$$\mathrm{INC}_T^\mu(R) = \{(a,b) \in E_1 \times E_2 | R(a,b) > J_\mu^T(a,b)\}$$

5. The μ-T-conditionalized relation of R, [8], is defined as: $R_C^{\mu-T}(a,b) = \mathrm{Min}(R(a,b), J_\mu^T(a,b))$

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$$= \begin{cases} R(a,b) & \text{if } T_R^{\mu}(a,b) \le \mu(b) \\ J_{\mu}^{T}(a,b) & \text{otherwise} \end{cases}$$

6. Given a fuzzy set μ: $E \to [0, 1]$ the fuzzy relation μ_2: $E_1 \times E_2 \to [0,1]$ is defined as $\mu_2(a,b) = \mu(b)$.

μ-T-inconditionality of fuzzy relations

Theorem 1
Let R be a fuzzy relation, let μ be a fuzzy set and let T be any continuous t-norm, then

$$J^T(R(a,b), J_{\mu}^{T}(a,b)) = J^T(T_R^{\mu}(a,b), \mu_2(a,b))$$

for all (a,b) in $E_1 \times E_2$

Proof
The proof is a consequence of Lemmata 1 to 7. (See Appendix)

Corollary
Given any continuous t-norm T, for all (a, b) in $E_1 \times E_2$, the distance $1 - J^T$ between a fuzzy relation R in the point (a, b) and its μ-T-conditionalized relation in (a, b) is the same as the distance $1 - J^T$ between $T_R^{\mu}(a,b)$ and $\mu_2(a,b)$.

μ-T-inconditionality measures of finite fuzzy relations

Definition
A T*-distance between two fuzzy relation R and R' is defined by

$$d_T(R, R') = \operatorname*{Sup}_{(a,b)\in E_1\times E_2} \{1 - J^T(R(a,b), R'(a,b))\}.$$

The measure of μ-T-inconditionality, M_T, of a fuzzy relation can be defined as

$$M_T(R) = d_T(R, J_{\mu}^{T}) = \operatorname*{Sup}_{(a,b)\in E_1\times E_2} \{1 - J^T(R(a,b), J_{\mu}^{T}(a,b))\}.$$

In other words:

For any continuous t-norm the distance d_T between a fuzzy relation R and its μ-T-conditionalized fuzzy relation is equal to the distance d_T between $T(\mu(a), R(a,b))$ and $\mu_2(a,b) = \mu(b)$.

For any continuous t-norm, a μ-T-inconditionality measure of fuzzy relation could be calculated as

$$\operatorname*{Sup}_{(a,b)\in E_1\times E_2} \{1 - J^T(R(a,b), J_{\mu}^{T}(a,b))\} \quad \text{or as}$$

$$\operatorname*{Sup}_{(a,b)\in E_1\times E_2} \{1 - J^T(T_R^{\mu}(a,b), \mu_2(a,b))\}.$$

Example:
Let μ be a fuzzy set on $E = \{a, b, c\}$ with membership degrees $\mu = \{0.2/a, 0.5/b, 0.8/c\}$.
Let R: $E \times E \to [0,1]$ be a fuzzy relation defined by

$$R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0.1 & 0.9 \\ 0.6 & 1 & 0.4 \\ 0.3 & 0.7 & 1 \end{pmatrix} \end{array}$$

Case 1: T = Min
When $T =$ Min, the residual relation J_{μ}^{Min} is represented by

$$\begin{pmatrix} 1 & 1 & 1 \\ 0.2 & 1 & 1 \\ 0.2 & 0.5 & 1 \end{pmatrix}.$$

..

The measure of μ-Min-inconditionality $M_{\text{Min}}(R)$ is computed as follows

$$\begin{aligned} M_{\text{Min}}(R) &= d_{\text{Min}}(R, J_{\mu}^{\text{Min}}) \\ &= \operatorname*{Sup}_{(a,b)\in E_1\times E_2} \{1 - J^{\text{Min}}(R(a,b), J_{\mu}^{\text{Min}}(a,b))\} \\ &= \text{Sup}\{1 - J^{\text{Min}}(0.6, 0.2), 1 - J^{\text{Min}}(0.3, 0.2), 1 \\ &\quad - J^{\text{Min}}(0.7, 0.5)\} \\ &= 1 - J^{\text{Min}}(0.6, 0.2) = 0.8 \end{aligned}$$

So R is not a μ-Min-conditional relation and the measure of μ-T-inconditionality is MMin(R) = 0.8.

Case 2: T = Product t-norm

J_{μ}^{Prod} is represented by $\begin{pmatrix} 1 & 1 & 1 \\ 0.4 & 1 & 1 \\ 0.25 & 0.625 & 1 \end{pmatrix}$.

The measure of μ-Prod-inconditionality $M_{\text{Prod}}(R)$ is computed as follows:

$$\begin{aligned} M_{\text{Prod}}(\ R\ ) &= d_{\text{Prod}}(R, J_{\mu}^{\text{Prod}}) \\ &= \operatorname*{Sup}_{(a,b)\in E_1\times E_2} \{1 - J^{\text{Prod}}(R(a,b), J_{\mu}^{\text{Prod}}(a,b))\} \\ &= \text{Sup}\{1 - J^{\text{Prod}}(0.6, 0.4), \\ &\quad 1 - J^{\text{Prod}}(0.3, 0.25), 1 - J^{\text{Prod}}(0.7, 0.625)\} \\ &= 1 - J^{\text{Prod}}(0.6, 0.4)\} = 0.333. \end{aligned}$$

So R is not a μ-Prod-conditional relation and the measure of μ-T-inconditionality is MProd(R) = 0.333.

Case 3: T = Ł ukasiewicz t-norm

J_{μ}^{W} is represented by $\begin{pmatrix} 1 & 1 & 1 \\ 0.7 & 1 & 1 \\ 0.4 & 0.7 & 1 \end{pmatrix}$, which contains R, so the measure of μ-W-inconditionality M_{Prod} (R) is computed as

$$M_W(R) = d_W(R, J_\mu^W) = \operatorname*{Sup}_{(a,b)\in E_1\times E_2} \{1 - J^W(R(a,b), J_\mu^W(a,b))\} = \mathrm{Sup}\{1-1,\ldots,1-1\} = 0$$

so R is a μ-W-conditional fuzzy relation.

μ-T-inconditionality measures of infinite fuzzy relations

Definition

$$M'_T(R) = \iint_{(a,b)\in E_1\times E_2} (1 - J^T(R(a,b), J_\mu^W(a,b)))\mathrm{da}\ \mathrm{db}$$

This new measure of μ-T-inconditionality is monotone. It is not based just on a point in which the supremum is reached, but it is based on the points in which the μ-T-conditional property does not hold. Observe that the expression in the integral is zero for all points on which the μ-T-conditional property holds.

Examples
Some operators are frequently used to make fuzzy inference. Fuzzy operators are fuzzy relations on the universe $E_1 \times E_2 = [0,1] \times [0,1]$.

The following examples show the evaluation of the measures M'_T for some implication operators. For all of them, it is taken the fuzzy set μ as the identity (i.e., as a function μ: $[0,1] \to [0,1]$ such that $\mu(x) = x$).
Table 1 shows the M'_T measures of the most used residual implication operators, S-implications, QM-implications and conjunctions for the t-norms minimun, product and Lukasiewicz:
The calculation of some of these measures is shown in what follows:

Example 1: Goguen implication
The Goguen implication is defined by

$$J^{\mathrm{Prod}}(x,y) = \begin{cases} 1 & \text{when } x \le y \\ y/x & \text{otherwise,} \end{cases}$$

so it is the residual operator J^{Prod} of the product t-norm. The values of this operator on $E_1 \times E_2 = [0,1] \times [0,1]$ are shown in Fig. 1.

It is known that this operator is Prod-conditional and W-conditional, but not Min-conditional. So it is computed the distance $1 - J^{\mathrm{Min}}$ between $\mathrm{Min}(\mu(x), J^{\mathrm{Prod}}(x,y))$ and $\mu(y)$. The graphical representation of the expression Min(x, Goguen(x,y)) is shown in Fig. 2.

The M'_{Min} measure of μ-Min-inconditionality of the Goguen implication operator is computed as:

$$\iint_{(x,y)\in E_1\times E_2} 1 - J^{\mathrm{Min}}(\mathrm{Min}(x, \mathrm{Goguen}(x,y)), y)\cdot \mathrm{d}x\ \cdot \mathrm{d}y$$
$$= \iint_{0\le x\le y\le 1} (1 - J^{\mathrm{Min}}(x,y))\cdot \mathrm{d}x\cdot \mathrm{d}y + \iint_{0\le x^2\le y<x\le 1} (1 - J^{\mathrm{Min}}(x,y))\mathrm{d}x\cdot \mathrm{d}y + \iint_{0\le y<x^2\le 1} (1 - J^{\mathrm{Min}}(\frac{y}{x}, y))\cdot \mathrm{d}x\cdot \mathrm{d}y$$
$$= \iint_{0\le x\le y\le 1} (1-1)\mathrm{d}\,\mathrm{d}y + \iint_{0\le x^2\le y<x\le 1} (1-y)\mathrm{d}x\,\mathrm{d}y$$

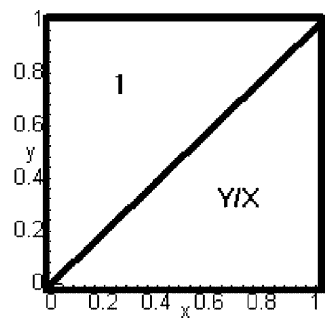

Fig. 1 The Goguen Implication

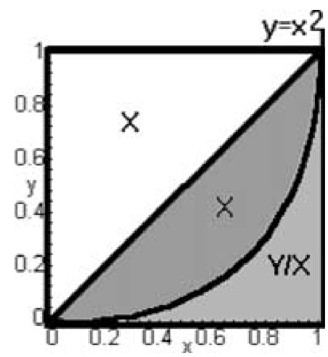

Fig. 2 Graphics of Min^{Id}_{Goguen}

Table 1 M′$_T$ measures of Id-T-inconditionality of some operators

Operator	T = Min	T = Prod	T = W
J^{Min}	0	0	0
J^{Prod}	$\frac{1}{3}$	0	0
Max(1-x, y)	$\frac{5}{24}$	$\frac{1}{30}$	0
1-x+xy	$\frac{1}{3}$	$\frac{3}{2} - 2\ln 2$	0
Min (x, y)	0	0	0
Prod (x, y) = xy	$\frac{1}{3}$	0	0
1 (x, y) = 1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$

$$+ \iint\limits_{0 \le y < x^2 \le 1} (1-y)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 \int_0^x (1-y)\mathrm{d}y\,\mathrm{d}x = \frac{1}{3}.$$

Example 2: Kleene–Dienes implication
The Kleene–Dienes implication operator is defined as $\mathrm{Max}(1-x, y)$. The graphical representation of the expression $\mathrm{Min}(x, \text{Kleene–Dienes}(x, y))$ is shown in Fig. 3. The M'_{Min} measure of μ-Min-inconditionality of the Kleene–Dienes implication is computed as follows

$$\iint\limits_{(a,b)\in E_1\times E_2} (1 - J^{\mathrm{Min}}(\mathrm{Min}(x, \mathrm{Max}(1-x, y)), y)\mathrm{d}x\,\mathrm{d}y$$

$$= \iint\limits_{0\le x\le y\le 1} (1 - J^{\mathrm{Min}}(x, y))\mathrm{d}x\,\mathrm{d}y$$

$$+ \iint\limits_{0\le y\le x\le \frac{1}{2}} (1 - J^{\mathrm{Min}}(x, y))\mathrm{d}x\,\mathrm{d}y$$

$$+ \iint\limits_{0\le y<\frac{1}{2}\le x\le 1} 1 - J^{\mathrm{Min}}(1-x, y)\mathrm{d}x\,\mathrm{d}y$$

$$= \iint\limits_{0\le x\le y\le 1} (1-1)\mathrm{d}x\,\mathrm{d}y + \iint\limits_{0\le y\le x\le \frac{1}{2}} (1-y)\mathrm{d}x\,\mathrm{d}y$$

$$+ \iint\limits_{0\le y<\frac{1}{2}\le x\le 1} (1-y)\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{1}{2}} \int_y^{1-y} (1-y)\mathrm{d}x\mathrm{d}y = \frac{5}{24}.$$

The graphical representation of the expression $\mathrm{Prod}(x, \mathrm{Max}(1-x, y))$ is shown in Fig. 4.

The M'_{Prod} measure of μ-Prod-inconditionality of the Kleene–Dienes implication is computed as follows

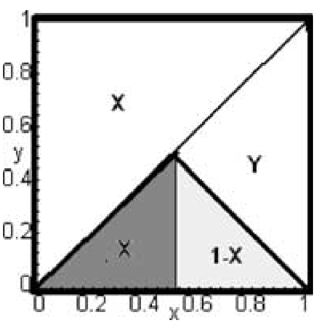

Fig. 3 Min(x, Kleene–Dienes(x, y))

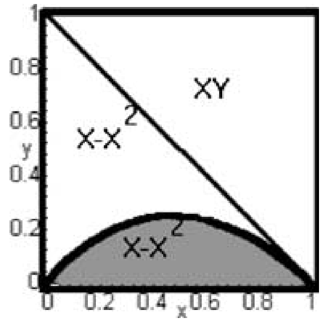

Fig. 4 Graphics of $Prod^{Id}_{Kleene-Dienes}$

$$\iint\limits_{(a,b)\in E_1\times E_2} (1 - J^{\mathrm{Prod}}(\mathrm{Prod}(x, \text{Kleene-Dienes}$$

$$(x, y)), y) \cdot \mathrm{d}x \cdot \mathrm{d}y$$

$$= \iint\limits_{0\le y\le x-x^2\le 1} (1 - J^{\mathrm{Prod}}(x \cdot \mathrm{Max}(1-x, y), y)) \cdot \mathrm{d}x \cdot \mathrm{d}y$$

$$= \iint\limits_{0\le x-x^2\le y\le 1} (1-1) \cdot \mathrm{d}x \cdot \mathrm{d}y$$

$$+ \iint\limits_{0\le y\le x-x^2\le 1} (1 - \frac{y}{x-x^2}) \cdot \mathrm{d}x \cdot \mathrm{d}y$$

$$= \int_0^1 \int_0^{x-x^2} (1 - \frac{y}{x-x^2}) \cdot \mathrm{d}y \cdot \mathrm{d}x = \frac{1}{30}.$$

Conclusion

This paper proposes two methods to study a degree for the μ-T-conditionality property of a fuzzy relation, in order to check whether the *modus ponens* is generalized when doing fuzzy inference.

The first method is based on computing a generalized distance between the fuzzy relation and the greatest μ-T-conditional relation contained in it. Another method consists in computing a generalized distance between $T(\mu(a), R(a, b))$ and $\mu(b)$ on all the points (a, b) in $E_1 \times E_2$ where the punctual property of μ-T-conditionality does not hold.

It is proven that for any continuous t-norm both methods give the same values when the generalized distance $1 - J^T$ is used.

There are defined two measures of inconditionality based on fuzzy relations distances, one for finite fuzzy relations an another one for infinite fuzzy relations.

Several examples are provided, especially for the most well known implication operators.

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Appendix

Preliminaries

1. A generalized metric space is a triplet $(E, \wp, m)$ where E is a set, $\wp = (A, S, \leq, e)$ is a commutative monoide with neutral element e, and m is an S-distance, which is a mapping m: $E \times E \to A$ such that:

1) $m(a,a) = e$, for all a in E
2) $m(a,c) \leq S(m(a,b), m(b,c))$, for all a, b, c in E (S-triangular inequality)

Let $T*$ be the dual t-conorm of a t-norm T, defined by

$T^*(x,y) = 1 - T(1-x, 1-y)$.

Let $\Im^* = ([0,1], T^*, \leq, 0)$ be a commutative ordered monoide with neutral element 0.
Given a relational structure (E, J), if J is a T-preorder then the function $d(a,b) = 1 - J^T(a,b)$ is a T^*-distance in the generalized metric space ([0, 1], $\Im^*$, d) [4].

2. The family of t-norms of a triangular t-norm T is the set of t-norms defined as: $T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x), \varphi(y))$, given any continuous, strictly increasing function φ: [0, 1] $\times$ [0, 1] $\to$ [0, 1] such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

Let $\varphi\mu$: $E \to [0,1]$ be the fuzzy set $(\varphi \circ \mu)$ defined by $\varphi\mu(a) = \varphi(\mu(a))$.
Let φR: $E_1 \times E_2 \to [0,1]$ be the fuzzy relation $(\varphi \circ R)$ defined by $\varphi R(a,b) = \varphi(R(a,b))$.

3. A t-norm T is an ordinal sum if there exist a finite or numerable collection of archimedean t-norms $\{T_i\colon\ i \in J\}$ and a collection of disjoint intervals $\{(a_i, b_i)\colon i \in J\}$ in [0, 1] such that

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) T_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & \text{if } (x,y) \in [a_i, b_i]^2 \\ \text{Min}(x,y) & \text{otherwise} \end{cases}$$

Remarks
Some necessary conditions for the point (a,b) of $E_1 \times E_2$ to be in $\text{INC}^\mu_T(R)$ are the following:
Condition (1): $R(a,b) > R_C^{\mu-T}(a,b) = J^T_\mu(a,b)$.
Condition (2): $T^\mu_R(a,b) = T(\mu(a), R(a,b) > \mu(b)$.
Condition (3): $\mu(a) > \mu(b)$.
Conditions (1) and (2) are held if and only if (a,b) is in $INC^\mu_T(R)$.

Lemma 1
Given a fuzzy relation R and given a fuzzy set μ, $J^{\text{Min}}(R(a,b), J^{\text{Min}}_\mu(a,b)) = J^{\text{Min}}(\text{Min}^\mu_R(a,b), \mu_2(a,b))$.

Proof

$$J^{\text{Min}}(R(a,b), J^{\text{Min}}_\mu(a,b)) = \begin{cases} 1 & \text{if } R(a,b) \leq J^{\text{Min}}_\mu(a,b) \\ J^{\text{Min}}_\mu(a,b) & \text{otherwise} \end{cases}$$

$$J^{\text{Min}}((\text{Min})^\mu_R(a,b), \mu_2(a,b)) = \begin{cases} 1 & \text{if } Min^\mu_R(a,b) \leq \mu(b) \\ \mu(b) & \text{otherwise} \end{cases}$$

If the point (a,b) is not in $\text{INC}^\mu_{\text{Min}}(R)$, then conditions (1) and (2) do not hold, so both expressions get the value 1.
If the point (a,b) is in $\text{INC}^\mu_{\text{Min}}(R)$, then:

$$\begin{aligned} J^{\text{Min}}(R(a,b), J^{Min}_\mu(a,b)) &= J^{Min}_\mu(a,b) && \text{by condition(1)} \\ &= \begin{cases} 1 & \text{if } \mu(a) \leq \mu(b) \\ \mu(b) & \text{if } \mu(a) > \mu(b) \end{cases} && \text{by condition(3)} \\ &= \mu(b) && \text{by condition(2)} \\ &= J^{\text{Min}}(\text{Min}^\mu_R(a,b), \mu(b)). && \blacksquare \end{aligned}$$

Lemma 2
Given a fuzzy relation R and a fuzzy set μ,

$J^{\text{Prod}}(R(a,b), J^{\text{Prod}}_\mu(a,b)) = J^{\text{Prod}}(Prod^\mu_R(a,b), \mu_2(a,b))$

Proof

$$J^{\text{Prod}}(R(a,b), J^{\text{Prod}}_\mu(a,b)) = \begin{cases} 1 & \text{if } R(a,b) \leq J^{\text{Prod}}_\mu(a,b) \\ J^{\frac{\text{Prod}_\mu(a,b)}{R(a,b)}} & \text{if } R(a,b) > J^{\text{Prod}}_\mu(a,b) \end{cases}$$

$$J^{\text{Prod}}(\text{Prod}^\mu_R(a,b), \mu_2(a,b)) = \begin{cases} 1 & \text{if Prod }^\mu_R(a,b) \leq \mu(b) \\ \frac{\mu(b)}{\text{Prod}^\mu_R(a,b)} & \text{if Prod }^\mu_R(a,b) > \mu(b) \end{cases}$$

If the point (a,b) is not in $\text{INC}^\mu_{\text{Prod}}$ (R), then conditions (1) and (2) do not hold, and both expressions are 1. If (a, b) is in $\text{INC}^\mu_{\text{Prod}}$ (R) then

$$J^{\text{Prod}}(R(a,b), J^{\text{Prod}}_\mu(a,b)) = \frac{J^{Prod}_\mu(a,b)}{R(a,b)} \quad \text{by conditions(1), (3)}$$

$$= \frac{\mu(b)}{\mu(a)R(a,b)}$$
$$= \frac{\mu(b)}{(Prod)_R^{\mu}(a,b)} \quad \text{by condition}(2)$$
$$= J^{\text{Prod}}(\text{Prod}_R^{\mu}(a,b), \mu(b))$$
$$= J^{\text{Prod}}(\text{Prod}_R^{\mu}(a,b), \mu_2(a,b)).$$

Lemma 3
Let W be the Lukasiewicz t-norm. Given a fuzzy relation R and given a fuzzy set μ,

$$J^W(R(a,b), J_\mu^W(a,b)) = J^W(W_R^\mu(a,b), \mu_2(a,b))$$

Proof
If the point (a,b) is not in $INC_W^\mu\ (R)$ then conditions (1) and (2) are not verified, so both expressions are 1. If (a,b) is in $\text{INC}_W^\mu\ (R)$, then:

$$J^W(R(a,b), J_\mu^W(a,b))$$
$$= \text{Min}(1, 1 - R(a,b) + J_\mu^W(a,b)) \quad \text{by condition}(1)$$
$$= 1 - R(a,b) + \text{Min}(1, 1 - \mu(a)$$
$$+ \mu(b)) \text{ by condition}(3)$$
$$= 1 - R(a,b) + 1 - \mu(a) + \mu(b)$$
$$= 1 - (\mu(a) + R(a,b) - 1) + \mu(b)$$
$$= 1 - \text{Max}(0, \mu(a) + R(a,b) - 1) + \mu(b)$$
$$= 1 - W(\mu(a), R(a,b)) + \mu(b) \quad \text{by condition}(2)$$
$$= \text{M in}(1, 1 - W_R^\mu(a,b) + \mu(b))$$
$$= J^W(W_R^\mu(a,b), \mu(b)) = J^W(W_R^\mu(a,b), \mu_2(a,b)) \quad \blacksquare$$

Lemma 4
Given a fuzzy relation R, and given a fuzzy set μ and a t-norm T_φ in the family of the product or the Lukasiewicz t-norm, it is verified that

$$J^{T_\varphi}(R(a,b), J_\mu^{T_\gamma}(a,b)) = J^{T_\varphi}((T_\varphi)_R^\mu(a,b), \mu_2(a,b))$$

Proof
Let T be the product or the Lukasiewicz t-norm. Applying Lemmas 2 and 3 to the fuzzy relation φR and to the fuzzy set $\varphi\mu$, it is deduced that

$$J^T(\varphi R(a,b), J_{\varphi\mu}^T(a,b)) = J^T(T_{\varphi R}^{\varphi\mu}(a,b), \varphi\mu_2(a,b)).$$

It is shown that this condition is held if and only if

$$J^{T_\varphi}(R(a,b), J_\mu^{T_\gamma}(a,b)) = J^{T_\varphi}((T_\varphi)_R^\mu(a,b), \mu_2(a,b)).$$

And then the result is proven for all t-norms in the family of the product and in the family of the Lukasiewicz t-norm. This proof is shown by the equivalence of the following equivalences

$$J^T(\varphi R(a,b), J_{\varphi\mu}^T(a,b)) = J^T(T_{\varphi R}^{\varphi\mu}(a,b), \varphi\mu_2(a,b))$$

by the definition of $J_{\varphi\mu}^T$ we have

$$J^T(\varphi R(a,b), J^T(\varphi\mu(a), \varphi\mu(b)))$$
$$= J^T(T_{\varphi R}^{\varphi\mu}(a,b), \varphi\mu_2(a,b))$$

by the definition of $T_{\varphi R}^{\varphi\mu}$

$$J^T(\varphi R(a,b), J^T(\varphi\mu(a), \varphi\mu(b)))$$
$$= J^T(T(\varphi\mu(a), \varphi R(a,b)), \varphi\mu_2(a,b))$$

adding $\varphi\varphi^{-1}$

$$J^T(\varphi R(a,b), \varphi\varphi^{-1}J^T(\varphi\mu(a), \varphi\mu(b)))$$
$$= J^T(\varphi\varphi^{-1}T(\varphi\mu(a), \varphi R(a,b)), \varphi\mu_2(a,b))$$

if and only if

$$J^T(\varphi R(a,b), \varphi J_\varphi^T(\mu(a), \mu(b)))$$
$$= J^T(\varphi T_\varphi(\mu(a), R(a,b)), \varphi\mu_2(a,b))$$

by the definition of $(T_\varphi)_R^\mu$

$$J^T(\varphi R(a,b), \varphi J_\varphi^T(\mu(a), \mu(b)))$$
$$= J^T(\varphi(T_\varphi)_R^\mu(a,b), \varphi\mu_2(a,b))$$

applying φ^{-1} to both sides

$$\varphi^{-1}J^T(\varphi R(a,b), \varphi J_\varphi^T(\mu(a), \mu(b)))$$
$$= \varphi^{-1}J^T(\varphi(T_\varphi)_R^\mu(a,b), \varphi\mu_2(a,b))$$

by the definition of J_φ^T

$$J_\varphi^T(R(a,b), J_\varphi^T(\mu(a), \mu(b)))$$
$$= J_\varphi^T((T_\varphi)_R^\mu(a,b), \mu_2(a,b))$$

by lemma 5

$$J^{T_\varphi}(R(a,b), J^{T_\varphi}(\mu(a), \mu(b)))$$
$$= J^{T_\varphi}((T_\varphi)_R^\mu(a,b), \mu_2(a,b))$$

and by the definition of $J_\mu^{T_\gamma}$

$$J^{T_\varphi}(R(a,b), J_\mu^{T_\gamma}(a,b))$$
$$= J^{T_\varphi}((T_\varphi)_R^\mu(a,b), \mu_2(a,b)).$$

Lemma 5
It is held that $J_\varphi^T = J^{T_\varphi}$

Proof

$$J_\varphi^T(x,y) = \varphi^{-1}J^T(\varphi(x), \varphi(y))$$
$$= \varphi^{-1}Sup\{\text{z} : T(\varphi(x), z) \le \varphi(y)\}$$
$$= \varphi^{-1}Sup\{\varphi(z) : T(\varphi(x), \varphi(z)) \le \varphi(y)\}$$
$$= \text{Sup}\{z : T(\varphi(x), \varphi(z)) \le \varphi(y)\}$$
$$= \text{Sup}\{z : \varphi^{-1}T(\varphi(x), \varphi(z)) \le y\}$$
$$= \text{Sup}\{z : T_\varphi(x,z) \le y\}$$
$$= J^{T_\varphi}(x,y). \quad \blacksquare$$

Lemma 6

Let T be an ordinal sum defined through a collection of archimedean t-norms $\{T_i\colon i \in J\}$ and a collection of disjoint intervals $\{(a_i, b_i) : i \in J\}$. The residuated operation of T is $J^T(x,y) = sup\{z/T(x,z) \le y\}$

$$= \begin{cases} 1 & \text{if } x \le y \\ y & \text{if } x > y \text{ and } (x,y) \notin [a_i,b_i] \text{for all } i \in J \\ a_i + (b_i - a_i) J_i^T\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & \text{if } x > y \text{ and } (x,y) \notin [a_i,b_i] \end{cases}$$

Proof

1) If $x \le y$, any residuated operation of a t-norm takes the value 1.
2) If $x > y$ and $(x, y) \notin [a_i, b_i]^2$, then $J^T(x,y) = y$. This is because
   2.1) If $x \notin [a_i, b_i]$, then $J^T(x,y) = \sup\{z : T(x,z) \le y\} = \sup\{z : Min(x,z) \le y\} = y$.
   2.2) If $x \in [a_i, b_i]$ and $y \notin [a_i, b_i]$, then $z = J^T(x, y)$ is not in $[a_i, b_i]$, because $y < x$, so $y < a_i$ and if $z = J^T(x,y)$ would be in $[a_i, b_i]$, then, by Lemma 1, $T(x,z)$ would also be in $[a_i, b_i]$ which contradicts $T(x,z) \le y$. As $z \notin [a_i, b_i]$, the t-norm must be the minimum and $J^T(x,y) = \sup\{z : T(x,z) \le y\} = \sup\{z : Min(x,z) \le y\} = y$.
3) If $x > y$ and $(x,y) \in [a_i, b_i]^2$, then $J^T(x,y) \in [a_i, b_i]$. This holds because if $z = J^T(x,y)$ would not be in $[a_i, b_i]$, then $J^T(x,y) = \sup\{z : T(x,z) \le y\} = \sup\{z : Min(x,z) \le y\} = y$, which contradicts with $y \in [a_i, b_i]$. So, in this case:

$$\begin{aligned} J^T(x,y) &= \sup\{z : T(x,z) \le y\} \\ &= \sup\left\{z/a_i + (b_i - a_i)T_i\left(\frac{x-a_i}{b_i a_i}, \frac{z-a_i}{b_i a_i}\right) \le y\right\} \\ &= \sup\left\{z/T_i\left(\frac{x-a_i}{b_i a_i}, \frac{z-a_i}{b_i a_i}\right) \le \frac{y-a_i}{b_i-a_i}\right\} \\ &= \left\{z/\frac{z-a_i}{b-a_i} = J^{Ti}\left(\frac{x-a_i}{b_i a_i}, \frac{y-a_i}{b_i a_i}\right)\right\} \\ &= a_i + (b_i - a_j) J^{Ti}\left(\frac{x-a_i}{b_i a_i}, \frac{z-a_i}{b_i a_i}\right). \end{aligned}$$

Definition

Let R be a fuzzy relation. The fuzzy relation R_{Ii} restricted from R to the interval $I_i = [a_i, b_i]$, is defined by:

$$R_{Ii}(a,b) = \frac{R(a,b) - a_i}{b_i - a_i} \quad .$$

Let μ be a fuzzy set. The fuzzy set μ_{Ii} restricted from μ to the interval $[a_i, b_i]$, is defined by:

$$\mu_{Ii}(a) = \frac{\mu(a) - a_i}{b_i - a_i} \quad .$$

Lemma 7

Let R be a fuzzy relation, μ be a fuzzy set and T be an ordinal sum, then

$$\begin{aligned} &J^T(R(a,b), J_\mu^T(a,b)) \\ &\quad = J^T(T_R^\mu(a,b), \mu_2(a,b)) \text{ for all } (a,b) \text{in} E_1 \times E_2. \end{aligned}$$

Proof

The proof follows from Sublemmas 1, 2, 3, 4, 5 and 6. ■

Sublemma 1

If $R(a, b) \le J_\mu^T(a, b)$ and if $T_R^\mu(a, b) \le \mu_2(a, b)$, then: $J^T(R(a,b), J_\mu^T(a,b)) = J^T(T_R^\mu(a,b), \mu_2(a,b))$.

Proof

By the properties of the residuated operation:

$$J^T(R(a,b), J_\mu^T(a,b)) = J^T(T_R^\mu(a,b), \mu_2(a,b)) = 1.$$

Note that in this case it is always held that $\mu(a) \le \mu(b)$. In the other cases, it is verified that $R(a, b) > J_\mu^T(a, b)$, so

$$T_R^\mu(a,b) > \mu_2(a,b) \text{and} \mu(a) > \mu(b). \quad ■$$

Sublemma 2

If $\mu(b) \notin [a_i, b_i]$, then

$$J^T(R(a,b), J_\mu^T(a,b)) = J^T(T_R^\mu(a,b), \mu_2(a,b)).$$

Proof

By Lemma 6, case 2.2 it holds that:

$$\begin{aligned} &J^T(R(a,b), J_\mu^T(a,b)) \\ &\quad = J^T(R(a,b), J^T(\mu(a), \mu(b)) \\ &\quad = J^T(R(a,b), \mu(b)) = \mu(b) \\ &\quad = J^T(T_R^\mu(a,b), \mu(b)) \\ &\quad = J^T(T_R^\mu(a,b), \mu_2(a,b)). \quad ■ \end{aligned}$$

Sublemma 3

If $\mu(a)$, $\mu(b) \in [a_i, b_i]$ and $R(a, b) \notin [a_i, b_i]$ then:

$$J^T(R(a,b), J_\mu^T(a,b)) = J^T(T_R^\mu(a,b), \mu_2(a,b)).$$

Proof
$T(\mu(a), R(a,b)) > \mu(b)$, so $R(a,b)) > \mu(b)$, but $R(a,b) \notin [a_i, b_i]$, so $a_i \le \mu(b) < \mu(a) \le b_i < R(a,b))$, and $\mathrm{Min}(\mu(a), R(a,b)) = \mu(a)$. By Lemma 6 it holds that:

$$J^T(R(a,b), J_\mu^T(a,b)) = J_\mu^T(a,b) = J^T(\mu(a), \mu(b))$$
$$= J^T(Min(\mu(a), R(a,b)), \mu(b)) = J^T(T(\mu(a), R(a,b)), \mu(b)) = J^T(T_R^\mu(a,b), \mu_2(a,b)). \blacksquare$$

Sublemma 4

If $R(a, b)$, $\mu(b)$ are in $[a_i, b_i]$ and $\mu(a) \notin [a_i, b_i]$, then

$$J^T(R(a,b), J_\mu^T(a,b)) = J^T(T_R^\mu(a,b), \mu_2(a,b)).$$

Proof
$T(\mu(a), R(a,b)) > \mu(b)$, so $R(a,b)) > \mu(b)$, and $a_i \le \mu(b) < R(a,b) \le b_i < \mu(a)$, so $\mathrm{Min}(\mu(\mathrm{a}), R(a,b)) = R(a,b)$.

$$J^T(R(a,b), J_\mu^T(a,b))$$
$$= J^T(R(a,b), J^T(\mu(a), \mu(b))$$
$$= J^T(R(a,b), \mu(b)) = J^T(Min(\mu(a), R(a,b)), \mu(b))$$
$$= J^T(T(\mu(a), R(a,b)), \mu(b))$$
$$= J^T(T_R^\mu(a,b), \mu_2(a,b)).$$

Sublemma 5

If $\mu(b) \in [a_i, b_i]$ and $\mu(a)$, $R(a, b)$ are not in $[a_i, b_i]$, then

$$J^T(R(a,b), J_\mu^T(a,b)) = J^T(T_R^\mu(a,b), \mu_2(a,b)).$$

Proof
Let T be an ordinal sum. If x, y $\in [a_i, b_i]$, then T(x, y) $\in [a_i, b_i]$. This is because T is monotonous, T(a_i, a_i) = a_i, T(b_i, b_i) = b_i , so $a_i \le T(x,y) \le b_i$.
As $\mu(a)$, R(a, b) are not in $[a_i, b_i]$, then T($\mu(a)$, R(a, b)) is not in $[a_i, b_i]$.

$$J^T(R(a,b), J_\mu^T(a,b))$$
$$= J^T(R(a,b), J^T(\mu(a), \mu(b))$$
$$= \mu(b) = J^T(T(\mu(a), R(a,b)), \mu(b))$$
$$= J^T(T_R^\mu(a,b), \mu_2(a,b)).$$

Sublemma 6
If μ(a), μ(b), R(a, b)$\in [a_i, b_i]$, then

$$J^T(R(a,b), J_\mu^T(a,b)) = J^T(T_R^\mu(a,b), \mu_2(a,b)).$$

Proof
Note 1: For archimedean t-norms it holds that J^T ($R(a, b)$, $J_\mu^T(a,b)) = J^T$ (T_R^μ (a, b), μ_2 (a, b)), so, for any t-norm T_i, the fuzzy relation R_{Ii} restricted to interval $[a_i, b_i]$ and the fuzzy set μ_{Ii} restricted to $[a_i, b_i]$ holds that

$$J^{Ti}\left(R_{Ii}(a,b), J_{\mu_{Ii}}^{Ti}(a,b)\right) = J^{Ti}((T_i)_{R_{Ii}}^{\mu_{Ii}}(a,b), \mu_{Ii}(b)).$$

Note 2: By lemma 5,

$$T(\mu(a), R(a,b))$$
$$= a_i + (b_i - a_i)T_i\left(\frac{\mu(a) - a_i}{b_i - a_i}, \frac{R(a,b) - a_i}{b_i - a_i}\right) \in [a_i, b_i].$$

By lemma 6, case 3, it holds that:

$$J^T(R(a,b), J_\mu^T(a,b))$$

$$= J^T\left(R(a,b), a_i + (b_i - a_i)J^{Ti}\left(\frac{\mu(a) - a_i}{b_i - a_i}, \frac{\mu(b) - a_i}{b_i - a_i}\right)\right)$$

$$= a_i + (b_i - a_i)J^{Ti}\left(\frac{R(a,b) - a_i}{b_i - a_i}, J^{Ti}\left(\frac{\mu(a) - a_i}{b_i - a_i}, \frac{\mu(b) - a_i}{b_i - a_i}\right)\right)$$

$$= a_i + (b_i - a_i)J^{Ti}\left(R_{Ii}(a,b), J_{\mu_{Ii}}^{Ti}(a,b)\right) \text{ See Note 1}$$

$$= a_i + (b_i - a_i)J^{Ti}((T_i)_{R_{Ii}}^{\mu_{Ii}}(a,b), \mu_{Ii}(b))$$

$$= a_i + (b_i - a_i)J^{Ti}(T_i\left(\frac{\mu(a) - a_i}{b_i - a_i}, \frac{R(a,b) - a_i}{b_i - a_i}\right), \frac{\mu(b) - a_i}{b_i - a_i})$$

$$= J^T(a_i + (b_i - a_i)T_i\left(\frac{\mu(a) - a_i}{b_i - a_i}, \frac{R(a,b) - a_i}{b_i - a_i}\right), \mu(b))$$

See Note 2

$$= J^T(T(\mu(a), R(a,b)), \mu(b))$$

$$= J^T(T_R^\mu(a,b), \mu_2(a,b)).$$