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On t-norms based measures of specificity

Luis Garmendia[a,*], Ronald R. Yager[b], Enric Trillas[c], Adela Salvador[c]

[a] *IBM Learning Services, Spain*
[b] *Iona College, USA*
[c] *Technical University of Madrid, Spain*

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Abstract

This paper gives a general expression for families of measures of specificity of a fuzzy set or a possibility distribution based on three t-norms and a negation. Other known measures of specificity are particular cases of this expression and new examples are provided. © 2002 Published by Elsevier Science B.V.

Keywords: Weak measures; t-Norms; Measures of specificity

1. Introduction

The concept of specificity provides a measure of the amount of information contained in a fuzzy set or possibility distribution by giving a degree for a fuzzy set to contain just one element. It is strongly related to the inverse of the cardinality of a set.

Let us remember that:

- Specificity measures were introduced by Yager [14, 15, 16, 18, 20, 21, 22, 23, 24] showing its usefulness as a measure of tranquility when making a decision. Yager introduced the specificity-correctness trade-off principle. The output information of expert systems and other knowledge-based systems should be both specific and correct to be useful. Yager suggested the use of specificity in default reasoning, in possibility qualified statements and data mining processes, giving several possible manifestations of this measure.
- Kacprzyk [8] described its use in a system for inductive learning.
- Dubois and Prade [4, 3] introduced the minimal specificity principle and showed the role of specificity in the theory of approximate reasoning.

* Corresponding author. c/Almansa 39, 3°B, 28039 Madrid, Spain.
E-mail address: luisgarmendia@es.ibm.com (L. Garmendia).

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- Higashi and Klir [7] introduced a closely related idea called *non-specificity*.
- The concept of granularity introduced by Zadeh [29] is correlated with the concept of specificity.

This paper proposes a new general definition to express the concept of specificity by using three t-norms and a negation. It is shown that other known formulas are particular cases of this general definition and new measures of specificity potentially useful in many applications are provided.

2. Preliminaries

Definition 1. A fuzzy set μ on X is *normal* if there exists an element $x_1 \in X$ such that $\mu(x_1) = a_1 = 1$.

Definition 2 (Measure of specificity). Let X be a set with elements $\{x_i\}$ and let $[0,1]^X$ be the class of fuzzy sets of X. A measure of specificity Sp is a function $\mathrm{Sp}: [0,1]^X \to [0,1]$ such that

1. $\mathrm{Sp}(\mu) = 1$ if and only if μ is a singleton ($\mu = \{x_1\}$).
2. $\mathrm{Sp}(\emptyset) = 0$.
3. If μ and η are normal fuzzy sets in X and $\mu \subset \eta$, then $\mathrm{Sp}(\mu) \geqslant \mathrm{Sp}(\eta)$.

The first condition imposes that the specificity is one (maximum value) only for crisp sets with just one element (singletons). The second condition assumes the minimum specificity for the null set. Other non-null fuzzy sets could also have specificity zero. The third condition requires that the specificity measure of a normal fuzzy set decreases when the membership degree of its elements increases.

If we would have to choose one element of a set of elements, and we have a fuzzy set with the degree of usefulness of each element, it is desirable to have a singleton or a high-specificity fuzzy set to be sure that our election is right.

Definition 3 (Weak measure of specificity). Let X be a set with elements $\{x_i\}$ and let $[0,1]^X$ be the class of fuzzy sets of X. A weak measure of specificity Sp is a function $\mathrm{Sp}: [0,1]^X \to [0,1]$ such that

1. $\mathrm{Sp}(\mu) = 1$ if μ is a singleton ($\mu = \{x_1\}$).
2. $\mathrm{Sp}(\emptyset) = 0$.
3. If μ and η are normal fuzzy sets in X and $\mu \subset \eta$, then $\mathrm{Sp}(\mu) \geqslant \mathrm{Sp}(\eta)$.

Note that the difference between a measure of specificity and a weak measure of specificity lies on axiom 1.

Specificity measures are not fuzzy measures [6] because they are not monotonous. The following definition of weak measure justifies the word ‘measures’, because specificity measures are weak measures.

Definition 4 (A weak concept of measure, Trillas and Alsina [13]). A measure of a characteristic k shown by the elements of a set E is made through a comparative relation like 'x shows the characteristic k less than y shows it' for any x, y in E.

Let us write '$x_k \preccurlyeq_k y$' to denote that relation and suppose that $\preccurlyeq_k$ is a preorder on E.
A function $m: E \to [0,1]$ is a $\preccurlyeq_k$-measure for E whenever:

1. $m(x_0) = 0$ if $x_0 \in E$ is minimal for $\preccurlyeq_k$.
2. $m(x_1) = 1$ if $x_1 \in E$ is maximal for $\preccurlyeq_k$.
3. If $x_k \preccurlyeq_k y$ then $m(x) \leqslant m(y)$.

Remarks. 1. Of course, fuzzy measures [6] are $\subseteq$-measures (monotonous measures), and fuzzy entropies [2] are $\leqslant_S$-measures, where $\subseteq$ is the contention and $\leqslant_S$ is the sharpened ordering.

2. Weak measures of specificity effectively measures the idea of how close is a fuzzy set from a singleton. So, a measure of specificity Sp is a weak measure where the set E is $[0,1]^X$, the characteristic k is the specificity of a fuzzy set, x_0 is the empty set, x_1 is a singleton and the preorder $\preccurlyeq_{\mathrm{Sp}}$ is defined as $\mu \preccurlyeq_{\mathrm{Sp}} \sigma \Leftrightarrow \mathrm{Sp}(\mu) \leqslant \mathrm{Sp}(\sigma)$.

3. Using the associative property of t-norms and t-conorms, generalized n-argument t-norms and t-conorms are easily defined [1].

3. t-norms and negation-based weak measure of specificity

Definition 5 (Measure of T-specificity $\mathrm{Sp_T}$). Let μ be a fuzzy set in a finite set X, and let a_i be the membership degree of the element x_i ($\mu(x_i) = a_i$). The membership degrees $a_i \in [0,1]$ are totally ordered with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. Let N be a negation [12], let T_1 and T_3 be any t-norms and let T_2^* an n-argument t-conorm. Let T be the quartet (T_1, N, T_2, T_3). Let $\{w_j\}$ be a weighting vector.

A *measure of T-specificity* Sp_T is an application $\mathrm{Sp}_T : [0,1]^X \to [0,1]$ defined by

$$\mathrm{Sp}_T(\mu) = T_1(a_1, N(T^*_{2\,j=2,\dots,n}\{T_3(a_j, w_j)\})).$$

Note: This formula represents the logical idea of 'one element' (represented by its membership degree a_1) 'and no others'. This first 'and' is implemented through the t-norm T_1. The negation of other elements is represented by a negation N of a general n-argument t-conorm T_2^* and the t-norm T_3.

Notation. Let us denote by $F(\mu)$ the function $T^*_{2\,j=2,\dots,n}\{T_3(a_j, w_j)\}$, so

$$\mathrm{Sp}_T(\mu) = T_1(a_1, N(T^*_{2\,j=2,\dots,n}\{T_3(a_j, w_j)\}) = T_1(a_1, N(F(\mu)))).$$

The three following lemmas prove that measures of T-specificity are weak measures of specificity.

Lemma 1. *If μ is a singleton then* $\mathrm{Sp}_T(\mu) = 1$.

Proof. Let μ be a singleton, then $a_1 = 1$ and $a_j = 0$ for all j: $2, \ldots, n$. So

$$\begin{aligned}\mathrm{Sp}_T(\mu) &= T_1(a_1, N(F(\mu))) = T_1(1, N(F(\mu))) = N(F(\mu)) = N(T^*_{2\, j=2,\ldots,n}\{T_3(0, w_j)\}) \\ &= N(T^*_{2\, j=2,\ldots,n}\{0, \ldots, 0\}) = N(0) = 1. \qquad \square\end{aligned}$$

Lemma 2. *The T-specificity of the empty set is zero* ($\mathrm{Sp}_T(\varnothing) = 0$).

Proof. $a_j = 0$ for all j, so $\mathrm{Sp}_T(\mu) = T(a_1, N(F(\mu))) = T(0, N(F(\mu))) = 0$. $\square$

Lemma 3. *If μ and η are normal fuzzy sets in X and $\mu \subset \eta$, then* $\mathrm{Sp}_T(\mu) \geqslant \mathrm{Sp}_T(\eta)$.

Proof. Let a_j and b_j, respectively, be the jth greatest membership degree of μ and η. $\mu \subset \eta$ so $a_j \leqslant b_j$ for all j and $T^*_{2\, j=2,\ldots,n}\{T_3(a_j, w_j)\} \leqslant T^*_{2\, j=2,\ldots,n}\{T_3(b_j, w_j)\}$. μ and η are normal, so $a_1 = b_1 = 1$, and

$$\begin{aligned}\mathrm{Sp}_T(\mu) &= T_1(a_1, N(F(\mu))) = T_1(1, N(F(\mu))) = N(F(\mu)) = N(T^*_{2\, j=2,\ldots,n}\{T_3(a_j, w_j)\}) \\ &\geqslant N(T^*_{2\, j=2,\ldots,n}\{T_3(b_j, w_j)\}) = T_1(1, N(F(\eta))) = T_1(a_1, N(F(\eta))) = \mathrm{Sp}_T(\eta). \qquad \square\end{aligned}$$

Theorem 1. *A measure of T-specificity is a weak measure of specificity.*

Proof. The proof follows from Lemmas 1–3. $\square$

Definition 6. A t-norm T is positive [10] when $T(x, y) = 0$ if and only if $x = 0$ or $y = 0$.

For example, the minimum t-norm and all t-norms in the family of the product t-norm are positive t-norms.

Lemma 4. *If T_3 is a positive* t-*norm*, *N is a strong negation and the weight w_2 is greater than zero*, *then the measure of T-specificity is a measure of specificity.*

Proof. Theorem 1 shows that a measure of T-specificity is a weak measure of specificity. It is proven that if $\mathrm{Sp}_T(\mu) = 1$ then μ is a singleton.

N is a strong negation, so $N(x) = 1$ if and only if $x = 0$.

Suppose that μ is not a singleton.

Case 1: $a_2 = 0$. Then $a_j = 0$ for all j: $2, \ldots, n$, and $N(F(\mu)) = N(0) = 1$. So $\mathrm{Sp}_T(\mu) = T_1(a_1, 1) = a_1$. μ is not a singleton so $\mathrm{Sp}_T(\mu) = a_1 \neq 1$.

Case 2: $a_2 \neq 0$. Then $T_3(a_2, w_2) > 0$, so $T^*_{2\, j=2,\ldots,n}\{T_3(a_j, w_j)\} > 0$, $N(T^*_{2\, j=2,\ldots,n}\{T_3(a_j, w_j)\}) < 1$ and $\mathrm{Sp}_T(\mu) = T_1(a_1, N(T^*_{2\, j=2,\ldots,n}\{T_3(a_j, w_j)\}) < 1)$. So if μ is not a singleton then $\mathrm{Sp}_T(\mu) \neq 1$. $\square$

Definition 7. A weak measure of specificity is *lower strict* when $\mathrm{Sp}(\mu)=0$ if and only if μ is the null set.

Lemma 5. *If both* t*-norms* T_1 *and* T_2 *are positive*, N *is a strong negation and* $w_j<1$ *for all* j: $2,\ldots,n$ *then the* T*-specificity measure is* lower strict.

Proof. Lemma 2 shows that if μ is the null set then $\mathrm{Sp}_T(\mu)=0$. It is proven that if $\mathrm{Sp}_T(\mu)=0$ then μ is the null set.

T_2 is positive, so the dual t-conorm $T_2^*(x_1,\ldots,x_n)=1$ if and only if exists j such that $x_j=1$. But $w_j<1$, so $T_3(a_j,w_j)\leqslant w_j<1$ for all j: $2,\ldots,n$. Thus $T^*_{2\,j=2,\ldots,n}\{T_3(a_j,w_j)\}<1$. N is a strong negation so $N(T^*_{2\,j=2,\ldots,n}\{T_3(a_j,w_j)\})=N(F(\mu))>0$.

Suppose that μ is not the null set, so $a_1>0$ and $\mathrm{Sp}_T(\mu)=T_1(a_1,N(F(\mu)))>0$. □

Corollary 1. *If* μ *and* η *are not null crisp subsets of* X *and* $\mathrm{card}(\mu)\geqslant\mathrm{card}(\eta)$ *then* $\mathrm{Sp}_T(\mu)\leqslant\mathrm{Sp}_T(\eta)$.

Proof. μ and η are crisp sets such that $a_j=1$ for j: $1,\ldots,m$ ($m=\mathrm{card}(\mu)$) and $a_j=0$ for j: $m+1,\ldots,n, b_j=1$ for j: $1,\ldots,s$ ($s=\mathrm{card}(\eta)$) and $b_j=0$ for j: $s+1,\ldots,n$, and $m\geqslant s$.

$$\begin{aligned}\mathrm{Sp}_T(\mu) &= T_1(a_1,N(F(\mu))) = T_1(1,N(F(\mu))) = N(F(\mu)) = N(T^*_{2\,j=2,\ldots,n}\{T_3(a_j,w_j)\})\\ &= N(T^*_{2\,j=2,\ldots,n}\{T_3(1,w_2),\ldots,T_3(1,w_m),T_3(0,w_{m+1}),\ldots,T_3(0,w_d)\})\\ &= N(T^*_{2\,j=2,\ldots,n}\{w_2,\ldots,w_m,0,\ldots,0\})\\ &\leqslant N(T^*_{2\,j=2,\ldots,n}\{w_2,\ldots,w_s,0,\ldots,0\} = \mathrm{Sp}_T(\eta)). \qquad \square\end{aligned}$$

Lemma 6. *If* μ *is a crisp set with cardinal* m, $1<m\leqslant n$, *the greatest weight is* w_M *and* T_2^* *is the* n*-argument* t*-conorm maximum then* $\mathrm{Sp}_T(\mu)=N(w_M)$.

Proof. μ is a crisp set such that $a_j=1$ for j: $1,\ldots,m$ ($m=\mathrm{card}(\mu)$) and $a_j=0$ for j: $m+1,\ldots,n$.

$$\begin{aligned}\mathrm{Sp}_T(\mu) &= T_1(a_1,N(F(\mu))) = T_1(1,N(F(\mu))) = N(F(\mu)) = N(T^*_{2\,j=2,\ldots,n}\{T_3(a_j,w_j)\})\\ &= N(\mathrm{Max}\{T_3(1,w_2),\ldots,T_3(1,w_m),T_3(0,w_{m+1}),\ldots,T_3(0,w_n)\})\\ &= N(\mathrm{Max}\{w_2,\ldots,w_m,0,\ldots,0\} = N(\mathrm{Max}\{w_2,\ldots,w_m\} = N(w_M))). \qquad \square\end{aligned}$$

Theorem 2. *If* $L=(\wedge,\vee,')$ *is a logic triplet based on a* t*-norm* $\wedge$, *its dual* t*-conorm* $\vee$ *and a negation* ' *then*

$$\mathrm{Sp}_L(\mu) = a_1 \wedge (a_2' \vee w_2') \wedge \cdots \wedge (a_n' \vee w_n') \text{ is a weak measure of specificity.}$$

Proof. It is shown that Sp_L is a particular case of measure of T-specificity with $T=(\wedge,{}',\wedge,\wedge)$ which are weak measures of specificity. Suppose that $T_1=T_2=T_3=\wedge$, and that $N={}'$.

$$\begin{aligned}\mathrm{Sp}_T(\mu) &= T_1(a_1, N(T^*_{2\,j=2,\dots,n}\{T_3(a_j,w_j)\}) = T_1(a_1, T_{2j=2,\dots,n}\{N(T_3(a_j,w_j))\})\\ &= T_1(a_1, T_{2j=2,\dots,n}\{T_3^*(N(a_j),N(w_j))\})\\ &= a_1\wedge(a_2'\vee w_2')\wedge\cdots\wedge(a_n'\vee w_n') = \mathrm{Sp}_L(\mu). \qquad \square\end{aligned}$$

This expression allows a new interpretation of weak specificity measures as $a_1\wedge P_2\wedge\cdots\wedge P_n$ where values P_j are penalties for elements $x_2,\dots,x_n$.

Corollary 2. *Let $L=(\wedge,\vee,{}')$ be a logical triplet based on a positive* t-*norm $\wedge$, its dual* t-*conorm $\vee$ and a negation ${}'$. If $w_2>0$ then*

$$\mathrm{Sp}_L(\mu) = a_1\wedge(a_2'\vee w_2')\wedge\cdots\wedge(a_n'\vee w_n') \text{ is a measure of specificity.}$$

Proof. The proof follows from Theorem 2 and Lemma 4. □

Definition 8. Let Sp and Sp^* be measures of specificity on the space X. Sp is *more critical* than Sp^* when their respective weights w_j and w_j^* verify $w_j \geqslant w_j^*$ for all j.

Definition 9. Let Sp and Sp^* be measures of specificity on the space X. Sp is *stricter* than Sp^*, denoted by $\mathrm{Sp}\leqslant\mathrm{Sp}^*$ [26], if for all fuzzy subsets μ of X $\mathrm{Sp}(\mu)\leqslant\mathrm{Sp}^*(\mu)$.

Definition 10. The *T-class of weak measures of specificity* is the set of measures Sp_T defined by the same t-norms and the same negation.

Lemma 7. *Let Sp_T and Sp_T^* be weak measures of specificity in the same T-class of weak measures of specificity. If Sp_T is more critical than Sp_T^* then Sp_T is stricter than Sp_T^*.*

Proof. Sp_T is more critical than Sp_T^*, so $w_j\geqslant w_j^*$ for all j. Thus $T_3(a_j,w_j)\geqslant T_3(a_j,w_j^*)$ for all j and $F(\mu)\geqslant F^*(\mu)$. So $N(F(\mu))\leqslant N(F^*(\mu))$ and $\mathrm{Sp}_T(\mu)\leqslant\mathrm{Sp}_T^*(\mu)$. □

Definition 11. A weak measure of specificity is *regular* [26] if for all constant fuzzy sets ($\mu_c(x)=c$ for all x) $\mathrm{Sp}_T(\mu_c)=0$.

4. Examples

Measures of T-specificity allow to obtain many different expressions of weak measures of specificity and measures of specificity of a fuzzy set or a possibility distribution in order to evaluate the usefulness of the information in many different environments. Measures of T-specificity provide a simple general formula that could be useful to implement any measure of weak specificity needed in applications.

It is shown that the most important known measures of specificity for a finite space are measures of T-specificity.

Example 1. Yager introduced [20] the linear measures of specificity on a finite space X as

$$\mathrm{Sp}(\mu) = a_1 - \sum_{j=2}^{n} w_j a_j,$$

where a_j is the jth greatest membership degree of μ and $\{w_j\}$ is a set of weights verifying:

1. $w_j \in [0,1]$.
2. $\sum_{j=2}^{n} w_j = 1$.
3. $w_j \geqslant w_i$ for all $1 < j < i$.

Theorem 3. *Linear measures of specificity are measures of T-specificity with $T = (W, N, W, Product)$ where W is the Lukasiewicz* t-*norm and N is the negation $N(x) = 1 - x$.*

Proof. Let T_1 and T_2 be the Lukasiewicz t-norm defined by $T_1(a,b) = \max\{0, a + b - 1\}$ and $T_2^*(a_1, \ldots, a_n) = \min\{1, a_1 + \cdots + a_n\}$.

$$\begin{aligned}
\mathrm{Sp}_T(\mu) &= T_1(a_1, N(T^*_{2\ j=2,\ldots,n}\{T_3(a_j, w_j)\})) = \mathrm{Max}\{0, a_1 + N(F(\mu)) - 1\} \\
&= \mathrm{Max}\{0, a_1 + (1 - F(\mu)) - 1\} = \mathrm{Max}\{0, a_1 - F(\mu)\} \\
&= \mathrm{Max}\{0, a_1 - T^*_{2\ j=2,\ldots,n}\{T_3(a_j, w_j)\}\} \\
&= \mathrm{Max}\left\{0, a_1 - \min\left\{1, \sum_{j=2}^{n} w_j a_j\right\}\right\} \qquad (1) \\
&= a_1 - \sum_{j=2}^{n} w_j a_j. \qquad (2)
\end{aligned}$$

It follows the explanation of the last equality:

(1) $a_j \leqslant 1 \Rightarrow \sum_{j=2}^{n} w_j a_j \leqslant \sum_{j=2}^{n} w_j 1 = \sum_{j=2}^{n} w_j = 1 \Rightarrow \min\{1, \sum_{j=2}^{n} w_j a_j\} = \sum_{j=2}^{n} w_j a_j$.
(2) $a_1 \geqslant a_j \Rightarrow \sum_{j=2}^{n} w_j a_j \leqslant \sum_{j=2}^{n} w_j a_1 = a_1 \sum_{j=2}^{n} w_j = a_1 \Rightarrow a_1 - \sum_{j=2}^{n} w_j a_j \geqslant 0 \Rightarrow \mathrm{Max}\{0, a_1 - \sum_{j=2}^{n} w_j a_j\} = a_1 - \sum_{j=2}^{n} w_j a_j$. □

Lemma 8. *Linear measures of specificity are measures of specificity.*

Proof. It follows from Theorem 3 that linear measures of specificity are measures of T-specificity and, from Theorem 1, they are also weak measures of specificity.

In the case of the linear T-measure of specificity, T_3 is the product, $N(x) = 1 - x$ is a strong negation and conditions 2 and 3 of the weights for linear measures of specificity imply that $w_2 > 0$, so the proof follows from Lemma 4. □

Properties (Yager [26]).

- Linear measures of specificity are regular.
- $\mathrm{Sp}(\mu) = a_1 - a_2$ is the strictest linear measure of specificity.
- The less stricter linear measure of specificity is

$$\mathrm{Sp}(\mu) = a_1 - \frac{1}{n-1} \sum_{j=2}^{n} a_j.$$

Corollary 3. *Yager's measure of specificity* [26] *on a finite space X defined by*

$$\mathrm{Sp}(\mu) = \int_0^{\alpha_{\max}} \frac{1}{\mathrm{Card}(\mu_\alpha)}\, \mathrm{d}\alpha,$$

is a measure of T-specificity.

Proof.

$$\int_0^{\alpha_{\max}} \frac{1}{\mathrm{Card}(\mu_\alpha)}\, \mathrm{d}\alpha$$

is a particular case of linear measure of specificity taking the weights as $w_2 = \frac{1}{2}$ and $w_j = 1/(j-1) - 1/j$ for all $j > 2$, so following Theorem 3 it is also a measure of T-specificity with $T = (W, N, W, \mathrm{Product})$. □

Example 2. Yager [26] defined the product measure of specificity for multi-criteria decision-making problems by $\mathrm{Sp}(\mu) = a_1 \prod_{j=2}^{n} (ka_j + (1 - a_j))$, where $k \in [0, 1)$.

This formula measures the existence of an element with membership degree one and all others with membership degree zero.

Theorem 4. $\mathrm{Sp}(\mu) = a_1 \prod_{j=2}^{n} (ka_j + (1 - a_j))$ *where $k \in [0, 1)$ is a measure of T-specificity with $T = (\mathrm{Prod}, N, \mathrm{Prod}, \mathrm{Prod})$ and $w_j = 1 - k$ for all j.*

Proof. If $T = (\mathrm{Prod}, N, \mathrm{Prod}, \mathrm{Prod})$ and $w_j = 1 - k$ for all j then:

$$\mathrm{Sp}_T(\mu) = T_1(a_1, N(T^*_{2\; j=2,\ldots,n}\{T_3(a_j, w_j)\})) = T_1(a_1, T_{2 j=2,\ldots,n}\{N(T_3(a_j, w_j))\})$$

$$= a_1 \prod_{j=2}^{n} N(a_j w_j) = a_1 \prod_{j=2}^{n} 1 - a_j w_j = a_1 \prod_{j=2}^{n} 1 - (1 - k) a_j$$

$$= a_1 \prod_{j=2}^{n} (ka_j + (1 - a_j)). \qquad \square$$

Corollary 4. *If $w_2 > 0$ then the product measure of specificity is a measure of specificity, and if $w_j < 1$ for all j then the product measure of specificity is a lower strict measure of specificity.*

Proof. It follows from Theorem 4 and Lemmas 4 and 5. □

Example 3. A more general example of product measures of specificity in the same product-class of measures of specificity is

$$\mathrm{Sp}(\mu) = a_1 \prod_{j=2}^{n} (1 - w_j a_j) \quad \text{where } w_j \in [0,1].$$

Corollary. *If $w_2 > 0$ then the general product measure of specificity is a measure of specificity, and if $w_j < 1$ then the general product measure of specificity is lower strict.*

Proof. It follows from Theorem 4 and Lemmas 4 and 5. □

Example 4 (Distance related measures of specificity). Another point of view for measures of specificity are distance-related measures of specificity. A fuzzy set μ of a set X with cardinal n can be seen as a vector of dimension n or as a point in $[0,1]^n$. Let E_i be the characteristic function of the singleton $(0, \ldots, 1^{(i)}, \ldots, 0)$, which can be seen as a collection of base vectors. The distance-related measure of specificity of a fuzzy set is defined through a negation operation of the closest distance of the fuzzy set with a singleton.

Let d_p be the p-euclidean distance defined by

$$d_p(\mu, \eta) = \left(\sum_{j=1}^{n} |a_i - b_i|^p \right)^{1-p}.$$

Yager shows [26] that the normalized metric $F(d_p(\mu,\eta)) = \min(1, d_p(\mu,\eta))$ is also a W-distance, it is, a distance satisfying the W-triangular unequality $(F(d_p(\mu,\eta)) \leqslant W(F(d_p(\mu,\sigma)), F(d_p(\sigma,\eta)))$ for all μ, σ, η in $[0,1]^X$) and defines the measure of specificity of a fuzzy set μ as

$$S_p(\mu) = 1 - \min_i (d(\mu, E_i)).$$

Note:

$$W_p^*(x_1, \ldots, x_n) = \min\left(1, \sqrt[p]{\sum_{j=1}^{n} x_j^p}\right)$$

is a t-conorm in the family of Lukasiewicz t-conorms because $W_p^* = \varphi^{-1} \circ W \circ \varphi \times \varphi$ with $\varphi(x) = x^p$.

Theorem 5. *Euclidean p-distances related measures of specificity of normal fuzzy sets are measures of T-specificity with $T=(T_1,N,W_p^*,T_3)$ and $w_j=1$ for all j, where T_1 and T_3 are any* t*-norms and N is the negation $N(x)=1-x$.*

Proof. μ is a normal fuzzy set, so $a_1=1$ and the closest singleton point is E_1 (see [26]). So

$$\begin{aligned}
\mathrm{Sp}_T(\mu) &= T_1(a_1, N({W^*_p}_{j=2,\ldots,n}\{T_3(a_j,w_j)\})) \\
&= T_1(1, N({W^*_p}_{j=2,\ldots,n}\{T_3(a_j,1)\})) \\
&= N({W^*_p}_{j=2,\ldots,n}\{a_j\}) \\
&= 1-\left(\min\left(1, \sqrt[p]{\sum_{j=2}^{n} a_j^p}\right)\right) \\
&= 1-(\min(1, \sqrt[p]{|1-1|+|a_2^p-0|+\cdots+|a_n^p-0|})) \\
&= 1-(\min(1, d_p(\mu,E_1))) \\
&= 1-\min_i(d_p(\mu,E_i)). \qquad \square
\end{aligned}$$

Theorem 6. *Let d_0 be the distance defined by $d_0(\mu,\eta)=\mathrm{Max}_{j=1,\ldots,n}(|a_j-b_j|)$. 0-distances related measures of specificity of normal fuzzy sets are measures of T-specificity with $T=(T_1,N,Minimum,T_3)$ and $w_j=1$ for all j, where T_1 and T_3 are any* t*-norms and N is the negation $N(x)=1-x$.*

Proof. μ is a normal fuzzy set, so $a_1=1$ and the closest singleton point is E_1. So

$$\begin{aligned}
\mathrm{Sp}_T(\mu) &= T_1\left(a_1, N\left(\overset{*}{\underset{j=2,\ldots,n}{\mathrm{Min}}}\{T_3(a_j,w_j)\}\right)\right) \\
&= T_1\left(1, N\left(\underset{j=2,\ldots,n}{\mathrm{Max}}\{T_3(a_j,1)\}\right)\right) \\
&= N\left(\underset{j=2,\ldots,n}{\mathrm{Max}}\{a_j\}\right) \\
&= 1-\underset{j=2,\ldots,n}{\mathrm{Max}}\{|a_j-0|\} \\
&= 1-\mathrm{Max}\{|1-1|,|a_2-0|,\ldots,|a_n-0|\} \\
&= 1-(\min(1, d_0(\mu,E_1))) \\
&= 1-\min_i(d_0(\mu,E_i)). \qquad \square
\end{aligned}$$

5. Conclusion

Given three t-norms, a negation and a set of weights is defined as measure of T-specificity, which is proven to be a weak measure of specificity. The measure of T-specificity formula expresses the logical idea of 'one element and no others'. The first t-norm T_1 represents this first 'and' and should not be the minimum t-norm in order to not lose information. This provides an easy way to build up weak measures of specificity and measures of specificity formulas that could be used in many different applications. Most used measures of specificity are shown to be a particular case of measures of T-specificity.

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