Editor-in-Chief
George J Klir

Vol: 35 No: 6
December 2006
ISSN: 0308-1079

INTERNATIONAL JOURNAL OF

GENERAL SYSTEMS

Taylor & Francis
Taylor & Francis Group

SUBSCRIPTION INFORMATION

International Journal of General Systems (USPS permit number 021281) is a peer-reviewed journal, published bimonthly in February, April, June, August, October and December by Taylor & Francis, 4 Park Square, Milton Park, Abingdon, Oxfordshire, United Kingdom OX14 4RN.

Annual Institutional Subscription, Volume 35, 2007

Print ISSN 0308-1079 $2480 £1892

A subscription to the print edition includes free access for any number of concurrent users across a local area network to the online edition. ISSN 1563-5104

For more information, visit our website:
http://www.tandf.co.uk/journals

For a complete and up-to-date guide to Taylor & Francis Group's journals and books publishing programmes, and details of advertising in our journals, visit our website:
http://www.tandf.co.uk

Periodicals postage paid at Champlain, NY, by US Mail Agent IMS of New York, 100 Walnut Street, Champlain, NY. **US Postmaster**: Please send address changes to gGEN, PO Box 1518, Champlain, NY 12919, USA.

Dollar rates apply to subscribers in all countries except the UK and the Republic of Ireland where the pound sterling price applies. *Subscribers in the EU have the option of paying in Euros.* All subscriptions are payable in advance and all rates include postage. Journals are sent by air to the USA, Canada, Mexico, India, Japan and Australasia. Subscriptions are entered on an annual basis, i.e., January to December. Payment may be made by sterling cheque, dollar cheque, international money order, National Giro, or credit card (Amex, Visa, Mastercard).

Ordering information:
USA/Canada: Taylor & Francis Inc., Journals Department, 325 Chestnut Street, 8th Floor, Philadelphia, PA 19106, USA. **EU/Rest of World:** T&F Customer Services, T&F Informa UK Ltd, Colchester, Essex CO3 3LP, United Kingdom.

Back Issues
Taylor & Francis retains a three year back issue stock of journals. Older volumes are held by our official stockists: Periodicals Service Company, 11 Main Street, Germantown, NY 12526, USA to whom all orders and enquiries should be addressed. Telephone: +1 518 537 4700; Fax: +1 518 537 5899; E-mail: psc@periodicals.com

Abstracting and indexing services:
International Journal of General Systems is currently listed in *Astrophysics Data System, Cambridge Scientific Abstracts, INSPEC, ISI, Mathematical Reviews, New Jour, SciBase* and *Zentralblatt Math.*

International Journal of General Systems, Vol. 35, No. 6, December 2006, 687–698

Taylor & Francis
Taylor & Francis Group

A t-norm based specificity for fuzzy sets on compact domains

L. GARMENDIA†*, R. R. YAGER‡, E. TRILLAS¶ and A. SALVADOR¶

†Deptartamento de Sistemas Informáticos y Programación, Facultad de Informática, Universidad Complutense de, Juan del Rosal 8, 28040 Madrid, Spain
‡Iona College, New Rochelle, NY, USA
¶Technical University of Madrid, Madrid, Spain

(Received 30 July 2004; in final form 3 April 2006)

An expression for measuring the specificity of fuzzy sets on continuous domains is introduced. This expression is based on t-norms, negations and the Choquet integral. It is also proved that the new expression satisfies the given axioms for measure of specificity. New examples are provided.

Keywords: Fuzzy measure; Specificity; Measure of specificity; Weak measure of specificity

1. Introduction

The concept of specificity introduces a measure of the amount of information contained in a fuzzy set by giving a degree of "containing just one element". This is strongly related with the inverse of the cardinality of a set.

If we would have to choose one element of a set of elements, and we have a fuzzy set with a degree of satisfaction of each element, it is desirable to have a singleton or a high specificity fuzzy set to make an election with tranquillity.

Some previous works study the measures of specificity of fuzzy sets on discrete domains (Yager 1990), but the measures of specificity on continuous domains deserves a deeper study.

Garmendia *et al.* (2003) uses a general expression for measures of specificity of fuzzy sets on finite domains using t-norms, t-conorms and negations. The general expression also allows to generate many measures of specificity using different fuzzy connectives, so it is possible to find the best measure of specificity of fuzzy sets in every environment or logic.

This paper gives a general expression to measure the specificity of fuzzy sets on continuous domains. The expression uses t-norms, negations, fuzzy measures and the Choquet integral. Some properties and also that other known formulas are particular cases of

*Corresponding author. Email: lgarmend@fdi.ucm.es. Website: www.fdi.ucm.es/profesor/lgarmend

International Journal of General Systems
ISSN 0308-1079 print/ISSN 1563-5104 online © 2006 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/03081070600867054

this general expression are shown. It is proved that this general expression verifies the axioms of measures of specificity (Yager 1998). The new measures of specificity are potentially useful in many applications.

Yager (1998) proposes a first expression for measuring the specificity of fuzzy sets on continuous domains that a particular case of the general expression given in this paper. Yager's first example uses a normalized Lebesgue measure and can be written using the new expression. Several examples of measures of specificity of fuzzy sets on the interval [0, 1] are given using several t-norms.

The new expression of measures of specificity of fuzzy sets on continuous domains can be used to generate different formulas of measure of specificity of fuzzy sets for each environment and for each application.

2. Preliminaries

DEFINITION 2.1. A binary operation T: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a t-norm (Schweizer and Sklar 1983), if it satisfies the following axioms:

1. $T(1, x) = x$;
2. $T(x, y) = T(y, x)$;
3. $T(x, T(y, z)) = T(T(x, y), z)$; and
4. if $x \leq x'$ and $y \leq y'$ then $T(x, y) \leq T(x', y')$.

A binary operation S: $[0, 1] \times [0, 1]$ is a t-conorm if it satisfies 2–4 and $S(0, x) = x$.

DEFINITION 2.2. A map N: $[0, 1] \rightarrow [0, 1]$ is a negation if it satisfies the following conditions:

1. $N(0) = 1$, $N(1) = 0$
2. N is non-increasing

A negation N is strong if $N(N(x)) = x$.

DEFINITION 2.3. A fuzzy set μ on X is normal if there exist an element $x_1 \in X$ such that $\mu(x_1) = 1$.

DEFINITION 2.4. Measure of specificity

Let X be a set and let $[0, 1]^X$ be the class of fuzzy sets on X. A measure of specificity (Yager 1990) is a function Sp: $[0, 1]^X \rightarrow [0, 1]$ such that:

1. $\mathrm{Sp}(\emptyset) = 0$.
2. $\mathrm{Sp}(\mu) = 1$ if and only if μ is a singleton.
3. If μ and η are normal fuzzy sets in X and $\mu \subset \eta$, then $\mathrm{Sp}(\mu) \geq \mathrm{Sp}(\eta)$.

The first condition assumes that the empty set have minimum specificity. However other not empty fuzzy sets could also have specificity zero.

The second condition imposes that only crisp sets with just one element (singletons) can have specificity one (the maximum specificity).

The third condition requires that the specificity measure of a normal fuzzy set decreases when the membership degrees of its elements are increased.

DEFINITION 2.5. Regular measure of specificity: A measure of specificity Sp is regular (Yager 1991) if $\mathrm{Sp}(X) = 0$.

DEFINITION 2.6. Weak measures of specificity: Let X be a set with elements $\{x_i\}$ and let $[0, 1]^X$ be the class of fuzzy sets of X. A weak measure of specificity Sp (Garmendia *et al.* 2003) is a function Sp: $[0, 1]^X \rightarrow [0, 1]$ such that:

1. $\mathrm{Sp}(\emptyset) = 0$,
2. $\mathrm{Sp}(\mu) = 1$ if μ is a singleton ($\mu = \{x_1\}$), and
3. if μ and η are normal fuzzy sets in X and $\mu \subset \eta$, then $\mathrm{Sp}(\mu) \geq \mathrm{Sp}(\eta)$.

The difference between a measure of specificity and a weak measure of specificity lies on axiom 2. Non-singletons fuzzy sets can have maximum weak specificity.

DEFINITION 2.7. Fuzzy measure (Grabisch *et al.* 2000):

Let $\wp$ be a family of subsets of a set X, with $\emptyset, X \in \wp$. A mapping M: $\wp \rightarrow [0, 1]$ is called a fuzzy measure if:

(1) $M(\emptyset) = 0$,
(2) $M(X) = 1$, and
(3) if $A, B \in \wp$ and $A \subseteq B$ then $M(A) \leq M(B)$.
The triple (X, $\wp$ and M) is a fuzzy measure space.
In Yager (1998) only fuzzy measures that satisfy the following condition are considered:
(4) $M(B) = 0$ if and only if B is the empty set or B is a singleton.

Note that condition 4 is needed in some proofs, but it is a technical condition and it is very difficult to translate into natural language.

The measures of specificity are not fuzzy measures because they are not monotone with respect to the inclusion of fuzzy sets. The following definition of fuzzy $\leq_k$-measure allow us to use the word "measure" to compute the specificity of fuzzy sets, because the measures of specificity are fuzzy $\leq_k$-measures.

DEFINITION 2.8. $\leq_k$-measure (Trillas and Alsina 1999):

A measure of a characteristic k shown by the elements of a set E is done through a comparative relation like "x shows the characteristic $k < y$ shows it" for any x, y in E.

Let's write "$x \leq_k y$" to denote that relation and suppose that $\leq_k$ is a preorder on E.

A function m: $E \rightarrow [0, 1]$ is a fuzzy $\leq_k$-measure for E if it satisfy the following conditions:

1. $m(x_0) = 0$ if $x_0 \in E$ is minimal for $\leq_k$.
2. $m(x_1) = 1$ if $x_1 \in E$ is maximal for $\leq_k$.
3. If $x \leq_k y$ then $m(x) \leq m(y)$.

Remarks.

1. Fuzzy measures are $\subseteq$ -measures (monotone measures with the inclusion preorder).
2. The entropy measures (De Luca and Termini 1972) for fuzzy sets are $\leq_S$-measures, where $\leq_S$ is the sharpened ordering.
3. The measure of specificity (Yager 1990) represents the idea of measuring how far is a fuzzy set from a singleton. So, a measure of specificity Sp is a fuzzy $\leq_k$-measure where the set E is $[0, 1]^X$; the characteristic k is the specificity of a fuzzy set; x_0 is the empty set (the only minimal set); x_1 is a singleton (the maximal sets are all singleton) and the preorder $\leq_{Sp}$ is defined as $\mu \leq_{Sp} \sigma \Leftrightarrow \mathrm{Sp}(\mu) \leq \mathrm{Sp}(\sigma)$.

DEFINITION 2.9. Choquet integral (Choquet 1953):

Let (X, $\wp$ and M) be a fuzzy measure space. Let f: $X \rightarrow [0, \infty]$ be a measurable function. The fuzzy integral of f with respect to a fuzzy measure M by the Choquet integral is:

$$(C)\int_X f\cdot \mathrm{d}M = (C)\int_X f(w)\cdot \mathrm{d}M(w) = \int_0^\infty M(f(x) > \alpha)\cdot \mathrm{d}\alpha.$$

The Choquet integral (Nguyen and Walker 1996) is an extension of the classical Lebesgue integral for nonclassical measures, such as fuzzy measures, which are not necessarily additive measures.

3. An expression for measuring the specificity of fuzzy sets under continuous domains

The axioms of measure of specificity (definition 2.4) and weak measure of specificity (definition 2.6) of fuzzy sets are given. This paper's goal is to provide expressions and formulas that satisfy the previous axiomatic definitions and that be used when it is useful to measure the amount of information contained in a fuzzy set on an continuous domain in order to make a decision.

A general expression for measures of specificity of fuzzy sets on continuous domains using a t-norm, a strong negation and a fuzzy measure is given and it is proved that the new expression satisfies the weak measures of specificity axioms. When the fuzzy measure verifies the condition 4, which is a technical, then the expression satisfies the axioms for measures of specificity.

Let A be a fuzzy set on an continuous universe X and let α_{sup} be the supremum of the membership degrees of A. Let (X, $\wp$ and M) be a fuzzy measure space (definition 2.7), such that the fuzzy measure M verifies that:

(4) $M(B) = 0$ if and only if B is the empty set or B is a singleton.

Let $A_\alpha \in \wp$ be the α-cut level set of A. Let T be a t-norm (definition 2.1) and let N be a strong negation (definition 2.2).

An expression for measuring the specificity of a fuzzy set A on an continuous domain is given as follows:

$$\mathrm{MS}(A) = T\left(\alpha_{\mathrm{sup}}, N\left(\int_0^{\alpha_{\mathrm{sup}}} M(A_\alpha)\cdot \mathrm{d}\alpha\right)\right)$$

where $\int_0^{\alpha_{\mathrm{sup}}}$ is a Choquet integral (definition 2.9).

LEMMA 3.1. If A is a normal fuzzy set then:

$$\mathrm{MS}(A) = N\left(\int_0^1 M(A_\alpha)\cdot \mathrm{d}\alpha\right).$$

Proof

$$\mathrm{MS}(A) = T\left(\alpha_{\sup}, N\left(\int_0^{\alpha_{\sup}} M(A_\alpha)\cdot \mathrm{d}\alpha\right)\right) = T\left(1, N\left(\int_0^1 M(A_\alpha)\cdot \mathrm{d}\alpha\right)\right)$$
$$= N\left(\int_0^1 M(A_\alpha)\cdot \mathrm{d}\alpha\right).$$

□

Note that if A is a classical non empty set then $\mathrm{MS}(A) = N\left(\int_0^1 M(A_\alpha)\cdot \mathrm{d}\alpha\right)$

LEMMA 3.2. If A and B are non empty classical sets and $M(A) \geq M(B)$ then $\mathrm{MS}(A) \leq \mathrm{MS}(B)$. The proof is trivial from the previous lemma.

THEOREM 3.3. The measure of specificity expression under continuous domains MS verifies the axioms of measures of specificity (definition 2.4).

Proof.

Axiom 1*:*

$$\mathrm{MS}(\varnothing) = T\left(0, N\left(\int_0^0 0\cdot \mathrm{d}\alpha\right)\right) = T(0,1) = 0$$

Axiom 2*:*

$$\mathrm{MS}(\{x\}) = T\left(1, N\left(\int_0^1 M(A_\alpha)\cdot \mathrm{d}\alpha\right)\right) = N\left(\int_0^1 0\cdot \mathrm{d}\alpha\right) = N(0) = 1,$$

and

$$\mathrm{MS}(A) = 1 \Rightarrow \alpha_{\sup} = 1 \text{ and } N\left(\int_0^1 M(A_\alpha)\cdot \mathrm{d}\alpha\right) = 1 \Rightarrow \alpha_{\sup} = 1 \text{ and } \int_0^1 M(A\alpha)\cdot \mathrm{d}\alpha = 0$$
$$\Rightarrow \alpha_{\sup} = 1 \text{ and } M(A_\alpha) = 0,$$

so by applying the condition 4 it is deduced that A is a singleton.

Axiom 3. If A and B are normal fuzzy sets and $A \subseteq B$ then $M(A_\alpha) \leq M(B_\alpha)$ for any α, so:

$$\mathrm{MS}(A) = N\left(\int_0^1 M(A_\alpha)\cdot \mathrm{d}\alpha\right) \geq N\left(\int_0^1 M(B_\alpha)\cdot \mathrm{d}\alpha\right) = \mathrm{MS}(B).$$

□

Note. If the condition 4 is not imposed to the fuzzy measure M, then MS is a weak measure of specificity (definition 2.6).

LEMMA 3.4. MS is a regular measure of specificity.

Proof

$$\mathrm{MS}(X) = T\left(1, N\left(\int_0^1 M(X_\alpha)\cdot \mathrm{d}\alpha\right)\right) = N\left(\int_0^1 1\cdot \mathrm{d}\alpha\right) = N(1) = 0. \qquad \square$$

4. Measure of specificity for fuzzy sets on continuous domains (Yager 1998)

Yager (1998) gives a first example of measure of specificity for a fuzzy set on an continuous domain. This paper shows that the same example can be written using the new proposed expression, the usual negation and the Łukasiewicz t-norm.

Let X be an continuous set (for example, a real interval). Let A be a fuzzy set on X and let A_α be its α-cut.

Yager (1998) proposes a measure of specificity on an continuous domain given as follows:

$$\mathrm{Sp}(A) = \int_0^{\alpha_{\max}} F(M(A_\alpha))\mathrm{d}\alpha$$

where $\alpha_{\max}$ is the maximum membership degree of A, M is a measure on X and F is a function F: [0, 1] → [0, 1] verifying:

(1) $F(0) = 1$,
(2) $F(1) = 0$, and
(3) if $x > y$ then $0 \leq F(x) \leq F(y) \leq 1$.

Example 4.1. Let X be the real interval [0, 1] and let M be the Lebesgue-Stieltjes measure defined as $M([a, b]) = b - a$. Let F be the function $F(z) = 1 - z$. Let A be the fuzzy set defined by:

$$A(x) = \begin{cases} 2x & 0 \leq x \leq 0.5 \\ -2x + 2 & 0.5 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The graphical representation of A is given in figure 1:

For any α, $A_\alpha = [\alpha/2, (2-\alpha)/2]$ and $M(A_\alpha) = ((2-\alpha)/2) - (\alpha/2) = 1 - \alpha$. As $\alpha_{\max} = 1$ then:

$$\mathrm{Sp}(A) = \int_0^1 F(M(A_\alpha))\mathrm{d}\alpha = \int_0^1 (1 - (1 - \alpha))\mathrm{d}\alpha = 0.5$$

Yager gives another new concept for measuring the specificity of fuzzy sets on continuous domains when X is the real interval $[a, b]$ and $F(z) = 1 - z$:

$$\mathrm{Sp}(A) = \int_0^{\alpha_{\max}} F(M(A_\alpha))\mathrm{d}\alpha = \int_0^{\alpha_{\max}} (1 - M(A_\alpha))\mathrm{d}\alpha = \alpha_{\max} - \int_0^{\alpha_{\max}} M(A_\alpha)\mathrm{d}\alpha.$$

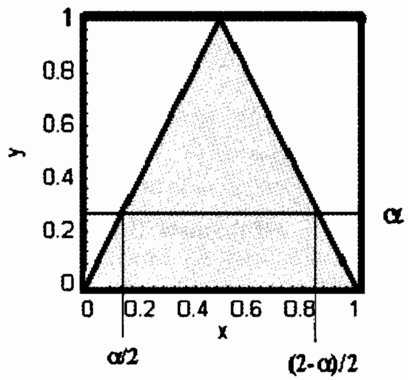


Figure 1. Fuzzy set $A(x) = \begin{cases} 2x & 0 \le x \le 0.5 \\ -2x+2 & 0.5 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$.

If M is the normalized Lebesgue measure $M(B) = \text{Length}(B)/(b - a)$ then

$$\text{Sp}(A) = \alpha_{\max} - \int_0^{\alpha_{\max}} M(A_\alpha)\text{d}\alpha = \alpha_{\max} - \frac{1}{b-a}\int_0^{\alpha_{\max}} \text{Length}(A_\alpha)\text{d}\alpha.$$

So, the expression $\int_0^{\alpha_{\max}} \text{Length}(A_\alpha)\text{d}\alpha$ can be interpreted as the area under the fuzzy set A, and the measure of specificity of a fuzzy set A on an interval $[a, b]$ can be given as

$$\alpha_{\max} - \frac{\text{area under } A}{b-a}.$$

5. The new expression generalises Yeager's measure of specificity of fuzzy sets on continuous domains (Yager 1998)

It is shown that the previous example 4.1 is a weak measure of specificity (definition 2.6) under an continuous domain when N is the negation $N(x) = 1 - x$, T is the Łukasiewicz t-norm defined by $T(x, y) = \max(0, x + y - 1)$, and M is the Lebesgue measure given by the length of an interval. Then

$$\text{MS}(A) = \max(0, \alpha_{\max} + N\left(\int_0^{\alpha_{\max}} M(A_\alpha)\cdot\text{d}\alpha\right) - 1$$

$$= \max\left(0, \alpha_{\max} + 1 - \int_0^{\alpha_{\max}} M(A_\alpha)\cdot\text{d}\alpha - 1\right)$$

$$= \max\left(0, \alpha_{\max} - \int_0^{\alpha_{\max}} M(A_\alpha)\cdot\text{d}\alpha\right) (M(A_\alpha) \text{ is always less or equal than one})$$

$$= \alpha_{\max} - \int_0^{\alpha_{\max}} M(A_\alpha)\cdot\text{d}\alpha = \int_0^{\alpha_{\max}} (1 - M(A_\alpha))\text{d}\alpha = \int_0^{\alpha_{\max}} F(M(A_\alpha))\text{d}\alpha = \text{Sp}(A).$$

Note. When the measure M is the length of an interval, it does not verify condition 4, hence the new given expression is a weak measure of specificity. For example, if

$$A(x) = \begin{cases} 1 & \text{if} \quad x = 0, \quad x = 0.25 \quad x = 0.5 \quad x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$M(A_1) = 0 \text{ and } M(A_0) = 1, \text{ then } \int_0^1 M(A_\alpha) \cdot \mathrm{d}\alpha = 0 \text{ and } \mathrm{Sp}(A) = 1 - \int_0^1 M(A_\alpha) \cdot \mathrm{d}\alpha = 1,$$

but A is not a singleton.

6. Examples

Example 6.1. To compute a weak measure of specificity of the fuzzy set

$$B(x) = \begin{cases} 0 & 0 \le x \le 0.25 \\ 4x - 1 & 0.25 \le x \le 0.5 \\ -4x + 3 & 0.5 \le x \le 0.75 \\ 0 & 0.75 \le x \le 1 \end{cases}$$

on the real interval [0, 1], it is necessary to compute its α-cut. which is graphically shown in the following figure 2:

For any α, $B_\alpha = [(\alpha + 1)/4, (3 - \alpha)/4]$.

If T is the Łukasiewicz t-norm, $N(x) = 1 - x$ and M is the Lebesgue measure then:

$$M(B_\alpha) = \frac{3 - \alpha}{4} - \frac{\alpha + 1}{4} = \frac{1 - \alpha}{2}.$$

As $\alpha_{\max} = 1$, it follows that

$$\mathrm{MS}(B) = 1 - \int_0^1 M(B_\alpha)\mathrm{d}\alpha = 1 - \int_0^1 \frac{1 - \alpha}{2}\mathrm{d}\alpha = 1 - \frac{1}{4} = \frac{3}{4}.$$

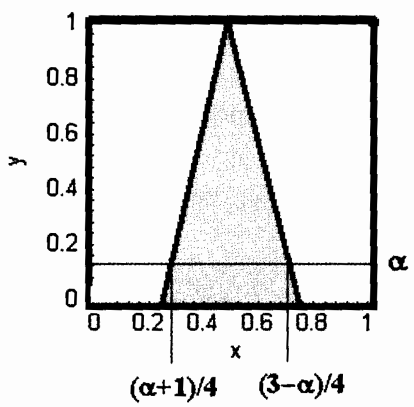


Figure 2. α-Cut of $B(x)$.

Example 6.2. To compute a weak measure of specificity of the fuzzy set

$$C(x) = \begin{cases} x & 0 \le x \le 0.5 \\ 1 - x & 0.5 \le x \le 1 \end{cases}$$

on the real interval [0, 1], it is necessary to compute its α-cut. For any α, $C_\alpha = [\alpha, 1 - \alpha]$ and $M(C_\alpha) = 1 - \alpha - \alpha = 1 - 2\alpha$ and so $\alpha_{\max} = 1/2$.

If $N(x) = 1 - x$ and T is the Łukasiewicz t-norm then

$$\mathrm{MS}(C) = \frac{1}{2} - \int_0^{1/2} M(C_\alpha)\mathrm{d}\alpha = \frac{1}{2} - \int_0^{1/2} (1 - 2\alpha)\cdot\mathrm{d}\alpha = \frac{1}{2} - \left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{4}.$$

If T = Prod, then

$$\mathrm{MS}(C) = \mathrm{Prod}\left(\alpha_{\max}, N\left(\int_0^{1/2} M(C_\alpha)\mathrm{d}\alpha\right)\right) = \frac{1}{2}\left(1 - \left[\frac{1}{2} - \frac{1}{4}\right]\right) = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8} = 0.375.$$

If T = Min, then

$$\mathrm{MS}(C) = \mathrm{Min}\left(\alpha_{\max}, N\left(\int_0^{1/2} M(C_\alpha)\mathrm{d}\alpha\right)\right) = \mathrm{Min}\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{3}{4} = 0.75.$$

Example 6.3. The following table 1 summarises several measures of specificity of five fuzzy sets defined on the unit interval when T is the minimum, Product or Łukasiewicz t-norm, $N(x) = 1 - x$ and M is the Lebesgue measure.

Note. When the fuzzy set is normal, the t-norm T is irrelevant. This is held because $a_1 = 1$ and so by lemma 3.1 it is held that

$$\mathrm{MS}(A) = N\int_0^1 M(A_\alpha)\cdot\mathrm{d}\alpha$$

Note that $B \subset A \subset E$, so $\mathrm{MS}(B) \ge \mathrm{MS}(A) \ge \mathrm{MS}(E)$, and as $D \subset C$ then $\mathrm{MS}(D) \ge \mathrm{MS}(C)$.

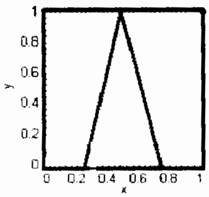
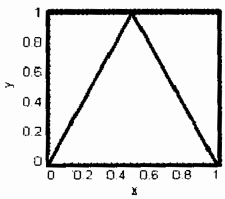
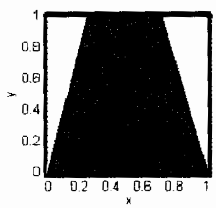
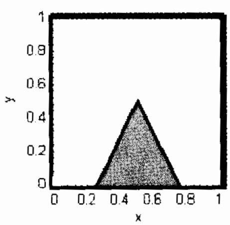
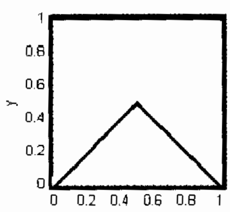
Example 6.4. Many other examples can be generated using different t-norms and negations. Some examples are given using the strong negation $N = 1 - x^2$.

If the t-norm T is the Łukasiewicz t-norm, then

$$\mathrm{MS}(C) = W\left(\alpha_{\max}, N\left(\int_0^{1/2} M(C_\alpha)\mathrm{d}\alpha\right)\right) = \frac{1}{2} + N\left(\int_0^{1/2} M(C_\alpha)\mathrm{d}\alpha\right) - 1 = \frac{1}{2} + \left(\frac{3}{4}\right)^2 - 1$$

$$= 0.0625.$$

Table 1. Examples of weak measures of specificity when $N(x) = 1 - x$, T = Min, Prod, W and M is the Lebesgue measure.

X=[0, 1]		*T=*	Łukasiewicz	*Product*	*Minimum*
B		$B(x) = \begin{cases} 0 & 0 \le x \le 0.25 \\ 4x - 1 & 0.25 \le x \le 0.5 \\ -4x + 3 & 0.5 \le x \le 0.75 \\ 0 & 0.75 \le x \le 1 \end{cases}$	0.75	0.75	0.75
A		$A(x) = \begin{cases} 2x & 0 \le x \le 0.5 \\ -2x + 2 & 0.5 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$	0.5	0.5	0.5
E		$E(x) = \begin{cases} 4x & 0 \le x \le \frac{1}{4} \\ 1 & \frac{1}{4} \le x \le \frac{3}{4} \\ 4 - 4x & \frac{3}{4} \le x \le 1 \end{cases}$	0.25	0.25	0.25
D		$D(x) = \begin{cases} 0 & 0 \le x \le 0.25 \\ 2x - \frac{1}{2} & 0.25 \le x \le 0.5 \\ \frac{3}{2} - 2x & 0.5 \le x \le 0.75 \\ 0 & 0.75 \le x \le 1 \end{cases}$	0.375	0.437	0.5
C		$C(x) = \begin{cases} x & 0 \le x \le 0.5 \\ 1 - x & 0.5 \le x \le 1 \end{cases}$	0.25	0.375	0.5

If T is the product t-norm, then

$$\mathrm{MS}(C) = \mathrm{Prod}\left(\alpha_{\max}, N\left(\int_0^{1/2} M(C_\alpha)\mathrm{d}\alpha\right)\right) = \frac{1}{2} * \left(\frac{3}{4}\right)^2 = 0.28125.$$

If T is the t-norm minimum, then

$$\mathrm{MS}(C) = \mathrm{Min}\left(\alpha_{\max}, N\left(\int_0^{1/2} M(C_\alpha)\mathrm{d}\alpha\right)\right) = \mathrm{Min}\left(\frac{1}{2}, \left(\frac{3}{4}\right)^2\right) = \mathrm{Min}(0.5, 0.5625) = 0.5.$$

Acknowledgements

This research is partially supported by the Spanish Ministry of Science and Technology, McyT and CAM.

References

A. De Luca and S. Termini, "A definition of a non-probabilistic entropy in the setting of fuzzy sets theory", *Inf. Control*, 20(4), pp. 301–312, 1972.
G. Choquet, "Theory of capacities", *Ann. Inst. Fourier*, 5, pp. 131–295, 1953.
L. Garmendia, R.R. Yager, E. Trillas and A. Salvador, "On t-norms based measures of specificity", *Fuzzy Sets Syst.*, 133(2), pp. 237–248, 2003.
M. Grabisch, T. Murofushi and M. Sugeno, *Fuzzy Measures and Integrals Theory and Applications*, Heidelberg: Physica-Verlag, 2000.
G.J. Klir, Z.Y. Wang and D. Harmanec, "Constructing fuzzy measures in expert systems", *Fuzzy Sets Syst.*, 92(2), pp. 251–264, 1997.
H.T. Nguyen and E.A. Walker, *A First Course in Fuzzy Logic*, Florida: CRC Press, 1996.
A. Ramer, J. Hiller and P. Diamond, "Total uncertainty revisited", *Int. J. Gen. Syst.*, 26(3), pp. 223–237, 1997.
D. Ralescu, "Cardinality, quantifiers, and the aggregation of fuzzy criteria", *Fuzzy Sets Syst.*, 69(3), pp. 355–365, 1995.
B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, New York: North-Holland, 1983.
E. Trillas and C. Alsina, "A reflection of what is a membership function", *Mathware Soft Comput. VI*, 2–3, pp. 201–215, 1999.
R.R. Yager, "On measures of specificity", in *Computational Intelligence: Soft Computing and Fuzzy-Neuro Integration with Applications*, O. Kaynak, L.A. Zadeh, B. Turksen and I.J. Rudas, Eds., Berlin: Springer-Verlag, 1988, pp. 94–113.
R.R. Yager, "Ordinal measures of specificity", *Int. J. Gen. Syst.*, 17, pp. 57–72, 1990.
R.R. Yager, "Measures of specificity of possibility distributions", *Proceedings of the Tenth NAFIPS Meeting*, Columbia, MO: University of Missouri, 1991, pp. 240–241.

Luis Garmendia received the Mathematics studies degree in 1995, with the speciality of Computers Science, by the Complutense University of Madrid. He received the PhD degree in Applied Mathematics in 2001 by the Technical University of Madrid. He worked in IBM Learning Services in Spain from 1998 to 2003, teaching database administration, SQL, application and system database performance in mainframe environments. He also worked at an IBM residency in Almaden Research Centre, California, at the International Technical Support Organization (ITSO) to write an IBM Redbook. He has been teaching computer programming and mathematics in several universities in Madrid: Alfonso X el Sabio, Carlos III, Technical University of Madrid, and actually at the Computing Faculty in the Complutense University of Madrid, where he also teaches PhD courses on approximate reasoning with uncertainty. Since 1995 he participate in most of the international conferences on Fuzzy Logic, writing 24 papers in proceedings, 9 papers in international journals, 2 books and two chapters of books. He is a member of FLAT (today EUSFLAT) since 1995. He has participated in 9 research projects and directed one research project on useful measures in fuzzy logic.

Ronald R. Yager received the undergraduate degree from the City College of New York and the PhD degree from the Polytechnic University of New York. He has served at the National Science Foundation as Program Director in the Information Sciences Program. He was a NASA/Stanford Visiting Fellow, as well as a Research Associate, at the University of California, Berkeley. He has served as a Lecturer at NATO Advanced Study Institutes. Currently, he is Director of the Machine Intelligence Institute and Professor of Information and Decision Technologies at Iona College, New Rochelle, NY. He has published over 500 articles and 15 books. In addition to his pioneering work in the area of fuzzy logic, he has made fundamental contributions in decision-making under uncertainty and the fusion of information. His current research interests include the development of technologies for a more intelligent Internet (E-Commerce, data mining, information retrieval) aggregation theory, decision-making under uncertainty, and higher order information fusion. Dr Yager is Editor-in-Chief of the *International Journal of Intelligent Systems*. He serves on the Editorial Board of a number of journals, including *The IEEE Transactions on Fuzzy Systems*, *Neural Networks*, *Data Mining and Knowledge Discovery*, *IEEE Intelligent Systems*, *Fuzzy Sets and Systems*, *The Journal of Approximate Reasoning* and *The International Journal of General Systems*. He is one of the cofounders of the Conference on Information Processing and the Management of Uncertainty (IPMU). He is a fellow of the New York Academy of Sciences and the Fuzzy Systems Association.

Enric Trillas received the PhD degree in Sciences in the University of Barcelona in 1972. From 1974 to 1985 he thought at the Technical University of Barcelona, where he took several academic responsibilities in the Computing and Architecture Faculties and as vice rector of the university extensions as vice rector of studies. At the late 1980s, he was president of the Spanish Council of Science Investigations (CSIC). In the 1990s he was the General Director of the INTA (Spanish Institute Of Space and Flying Techniques and General Secretary of the Spanish Plan of Investigations and Developments. Since 1990 he is a distinguished Professor at the Computing Science and Artificial Intelligence Department in the Faculty of Computing at the Technical University of Madrid. He is considered to be the person that introduced the research on fuzzy logic in Spain. He is author of many publications, he has directed 16 PhD and more than 20 projects. He is usual collaborator on international conferences, scientific committees, with many medals and awards, including the three most recognised awards on fuzzy logic: Pioneer Award of the European Society For Fuzzy Logic and Technologies (EUSFLAT), Fellow of the International Fuzzy System Association (IFSA) and the most recent Fuzzy Systems Pioneer Award.

Adela Salvador received the mathematics studies degree in 1969, in the Complutense University of Madrid. She received the PhD degree in 1988 in the Alcala de Henares University, and she was a pioneer on fuzzy algebra research. She was teaching mathematics in the Complutense University of Madrid from 1969 to 1971 and pre-universitary teacher mathematics from 1971 to 1990. During this period she wrote 59 books on how to teach mathematics and also several biographies of mathematician women. She is teaching Applied Mathematics in the Technical University of Madrid since 1990. She is a member of FLAT (today EUSFLAT) since 1995. She has written 70 papers in conference proceedings and international journals, and she has collaborated on the organization of 21 congresses. He has participated in 30 research projects.