Abstract

We build an order 2 scheme to integrate a fractional Dirac equation and we present the analysis of the convergence.

Keywords: fractional equation, Dirac equation, numerical analysis.

1 Introduction

Lately, the fractional Dirac equation has been considered as an interpolation from the inside between the classical Heat Equation and the Wave Equation [1, 2]. In this work we build a suitable numerical scheme to simulate a one-dimensional fractional Dirac equation of the form:

\[ \gamma^0 \partial_t^\alpha \psi(t, x) + \gamma^1 \partial_x \psi(t, x) = 0, \]

where \( \partial_t^\alpha \) is a fractional derivative \((0 < \alpha < 1)\) with respect to the variable \( t \), \( \psi \) is a two-component complex spinor, and

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

In order to build our numerical scheme, we start using the ideas presented in [3] for a system that depends on a single variable, we will then extend the ideas to the fractional equation. We start considering a fractional equation of the form:

\[ d^\alpha y(t) = f(t), \]

where \( d^\alpha \) is the fractional Caputo derivative of order \( \alpha \in (0, 1) \) with lower limit 0. We have that \( d^\alpha = d(J_{1-\alpha}) \) which means that \( d(J_{1-\alpha}y) = f \), where the fractional integral
of order $\alpha$ is defined by:

$$J_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) \, d\tau.$$  

(4)

On the other hand

$$dy(t) = J_{1-\alpha}^{-1} f(t) = d(J_\alpha f(t)).$$

From here, we have

$$d\left(y - J_\alpha f(t)\right) = 0$$

which supposes finally that

$$y(t) = y(0) + J_\alpha f(t).$$  

(5)

Let us define for a given time $t$, a time step $\Delta t$ such that $t = n\Delta t$. We then define the times $t_k \equiv k\Delta t$ for any $k \in \mathbb{N}$, and the initial time $t_0 \equiv 0$. We can split the integral into a sum of the form

$$\int_0^{t_n} (t_n - \tau)^{\alpha - 1} f(\tau) \, d\tau = \sum_{k=0}^{n-1} I_k,$$  

(6)

with

$$I_k \equiv \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha - 1} f(\tau) \, d\tau.$$  

(7)

In order to build numerical schemes, we will approximate $I_k$ by some quadrature formula.

1.1 First order schemes

The integral $I_{n-1}$ is an improper one, since the kernel $(t_n - \tau)^{\alpha - 1}$ is singular at the upper limit $t_n$, and we cannot, thus, use a standard quadrature formula. To solve this problem we will follow the idea considered in [3] and approximate $f$ by a constant value:

$$I_k^0 = \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha - 1} f(t_k) \, d\tau$$

$$= f(t_k) \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha - 1} \, d\tau$$

$$= \frac{1}{\alpha} f(t_k) \left( [t_n - t_{k+1}]^\alpha - [t_n - t_k]^\alpha \right)$$

$$= \frac{1}{\alpha} f(t_k) \left( t_n^\alpha - t_{k+1}^\alpha - t_n^\alpha + t_{n-k}^\alpha \right).$$  

(8)

Let us compute the truncation error that arises if we replace $I_k$ by $I_k^0$:

$$\varepsilon_k^0 \equiv I_k - I_k^0$$

$$= \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha - 1} [f(\tau) - f(t_k)] \, d\tau.$$  

(9)
We perform the change of variable $\tau = t_k + s\Delta t$, and we obtain
\[
\varepsilon_k^0 = \int_0^1 (t_n - t_k + s\Delta t)^{\alpha - 1}[f(t_k + s\Delta t) - f(t_k)]\Delta t \, ds .
\] (10)

The Taylor expansion of $f(t_k + s\Delta t)$ around $t_k$, assuming $f$ is at least $C^1$, gives us
\[
\varepsilon_k^0 = \int_0^1 (t_n - t_k + s\Delta t)^{\alpha - 1}f'(z(s)) s\Delta t^2 \, ds = f'(z_k) \int_0^1 (t_{n-k} + s\Delta t)^{\alpha - 1}s\Delta t^2 \, ds = \frac{\Delta t}{\alpha} \left[ t_{n-k+1}^{\alpha+1} - t_{n-k}^{\alpha+1} \right] f'(z_k) ,
\] (11)

with $z(s) \in [t_k, t_k + s\Delta t]$ and $z_k \in [t_k, t_{k+1}]$. It is important to stress that this error expression if bounded for all $k$, provided $f'(z_k)$ is bounded. The error is apparently of order $\Delta t$, but we may see that it is in fact of order $\Delta t^2$: if we perform a Taylor expansion of (11) around $t_{n-k}$, we have cancellation of the leading terms and obtain:
\[
\varepsilon_k^0 = \frac{\Delta t^2}{2} \tilde{z}_k^{\alpha-1}f'\left(\tilde{z}_k\right) ,
\] (12)

for some $\tilde{z}_k \in [t_k, t_{k+1}]$. Contrarily to what happened with (11), this is not necessarily bounded for all $k$, even if $f'$ is bounded, since for $k = 0$ we have $\tilde{z}_0 \in [0, \Delta t]$ and thus $\tilde{z}_k^{\alpha-1}$ might result unbounded. We have to keep in mind that this is just for the sake of formally computing the order and that the error is effectively bounded by expression (11). With all this, we may consider the numerical scheme given by:
\[
y_n = y_0 + \frac{\Delta t}{\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} f(t_k) \frac{t_{n-k-1}^\alpha - t_{n-k}^\alpha}{\Delta t}
\] (13)

with the corresponding local truncation error:
\[
E_n^0 = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \varepsilon_k^0
= \frac{1}{\Gamma(\alpha)} \Delta t \sum_{k=0}^{n-1} \frac{t_{n-k+1}^{\alpha+1} - t_{n-k}^{\alpha+1}}{(\alpha + 1)\Delta t} f'(z_k)
= \frac{1}{\Gamma(\alpha + 1)} f'(z_n) \sum_{k=0}^{n-1} \Delta t \left[ t_{n-k+1}^{\alpha+1} - t_{n-k}^{\alpha+1} \right] / (\alpha + 1)\Delta t
\] (14)

for some $z_n \in [0, t_n]$. Or, using (12) instead of (11):
\[
E_n^0 = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \frac{\Delta t^2}{2} \tilde{z}_k^{\alpha-1}f'\left(\tilde{z}_k\right)
= \frac{1}{2\Gamma(\alpha)} \tilde{z}_n^{\alpha-1}f'\left(\tilde{z}_n\right) t_n\Delta t ,
\] (15)
for some $\tilde{z}_n \in [0, t_n]$. This second expression shows that the local error is formally of first order in $\Delta t$.

We could have also taken the value of $f$ at the upper limit instead of at the lower one, and approximate $I_k$ by

$$I_k^1 = \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha-1} f(t_{k+1}) \, d\tau = \frac{1}{\alpha} f(t_{k+1}) \left( t_{n-k-1}^{\alpha} - t_{n-k}^{\alpha} \right).$$  \tag{16}$$

Repeating for this case the previous steps, we obtain the corresponding scheme given by

$$y_n = y_0 + \frac{\Delta t}{\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} f(t_{k+1}) - \frac{t_{n-k-1}^{\alpha} - t_{n-k}^{\alpha}}{\Delta t}.$$  \tag{17}$$

As before, we have two expressions for the local truncation error:

$$E_n^1 = -\frac{1}{\Gamma(\alpha + 1)} f'(z_n) \sum_{k=0}^{n-1} \Delta t \left[ t_{n-k}^{\alpha} - \frac{t_{n-k-1}^{\alpha} - t_{n-k}^{\alpha}}{(\alpha + 1)\Delta t} \right],$$  \tag{18}$$

$$E_n^1 = -\frac{1}{2\Gamma(\alpha)} \tilde{z}_n^{\alpha-1} f'(z_n) t_n \Delta t,$$  \tag{19}$$

for some $z_n$ and $\tilde{z}_n$ both in $[0, t_n]$.

### 1.2 Second order scheme

Both previous schemes are of first order, and it is clear that taking their average we can gain an order of accuracy, as is also usually the case in finite differences. Combining both schemes, we have

$$y_n = y_0 + \frac{\Delta t}{\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} \frac{f(t_{k+1}) + f(t_k)}{2} \frac{t_{n-k-1}^{\alpha} - t_{n-k}^{\alpha}}{\Delta t}.$$  \tag{20}$$

To obtain the local truncation error, we need the corresponding Taylor expansions, now with an extra order. We have:

$$E_n = -\frac{\Delta t^2}{2\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} \left[ f'(t_k) \left( \frac{t_{n-k-1}^{\alpha} - t_{n-k}^{\alpha}}{\Delta t} - \frac{t_{n-k}^{\alpha+1} - 2t_{n-k}^{\alpha} + t_{n-k-1}^{\alpha}}{(\alpha + 1)\Delta t^2} \right) + f''(z_k) \left( \frac{t_{n-k-1}^{\alpha+2} + t_{n-k}^{\alpha+2}}{2} - \frac{t_{n-k+1}^{\alpha+1} + t_{n-k}^{\alpha+1}}{(\alpha + 1)\Delta t} + \frac{t_{n-k}^{\alpha+2} + t_{n-k-1}^{\alpha+2}}{(\alpha + 1)(\alpha + 2)\Delta t} \right) \right],$$  \tag{21}$$

with $z_k \in [t_k, t_{k+1}]$. This is a bounded expression, whenever $f'$ and $f''$ are bounded. In order to see that this error is formally of order $\Delta t^2$, we perform a Taylor expansion about $t_{n-k}$. We have finally:

$$E_n = \frac{\Delta t^2}{6\Gamma(\alpha)} t_n \left[ 3(\alpha - 1)z_n^{\alpha-2} f'(z_n) - z_n^{\alpha-1} f''(z_n) \right],$$  \tag{22}$$

and $z_n \in [0, t_n]$.  

4
2 Numerical scheme for the fractional Dirac equation

Although we may build a numerical scheme for the fractional Dirac equation using any scheme of the previous section, our aim is to build a second order scheme.

2.1 Second order scheme

We use the preceding second order scheme to represent the fractional time derivative of the solution and a second order scheme for the spatial derivative. We start expressing the Dirac equation in a form similar to (3):

\[ \gamma^0 \partial_t^\alpha \psi(t, x) + \gamma^1 \partial_x \psi(t, x) = 0 \iff \partial_t^\alpha \psi(t, x) = -\gamma^0 \gamma^1 \partial_x \psi(t, x) \]  

(23)

and from here

\[ \psi(t_n, x) = \psi(0, x) - \gamma^0 \gamma^1 \frac{1}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - \tau)^{\alpha-1} \psi_x(\tau, x) \, d\tau. \]  

(24)

We discretize now both time and space. We will denote the spatial index by subscripts and the temporal one by superscripts: we have for instance \( \psi^n_l = \psi(t_n, x_l) = \psi(n \Delta t, l \Delta x) \). We start replacing the integral by the second order quadrature formula of the previous section:

\[ \psi^n_l = \psi^0_l - \frac{\Delta t \gamma^0 \gamma^1}{2 \Gamma(\alpha + 1)} \sum_{k=0}^{n-1} \left( \frac{(\psi^1_{x})^k+1_1 + (\psi^1_{x})^k_k}{2} \right) \frac{t_{n-k-1}^\alpha - t_{n-k}^\alpha}{\Delta t} + E_n, \]  

(25)

where \( E_n \) is the error given in the previous section, replacing \( f' \) by \( -\gamma^0 \gamma^1 \psi_{xt} \) and \( f'' \) by \( -\gamma^0 \gamma^1 \psi_{xtt} \). Taking into account now that the centered, second order, finite difference satisfies

\[ (\psi_x)^n_l = \frac{\psi^n_{l+1} - \psi^n_{l-1}}{2 \Delta x} - \frac{\Delta x^2}{6} \psi_{xxx}(t_n, \tilde{x}_l) \]  

(26)

for some \( \tilde{x}_l \in [x_{l-1}, x_{l+1}] \), we obtain:

\[ \psi^n_l = \psi^0_l - \frac{\Delta t \gamma^0 \gamma^1}{2 \Gamma(\alpha + 1)} \sum_{k=0}^{n-1} \left( \frac{(\psi^1_{x})^k+1_1 + (\psi^1_{x})^k_k}{2} \right) \frac{t_{n-k-1}^\alpha - t_{n-k}^\alpha}{\Delta t} + E_n + \frac{\Delta t \Delta x^2 \gamma^1}{6 \Gamma(\alpha + 1)} \sum_{k=0}^{n-1} \psi_{xxx}(\tilde{t}_k, \tilde{x}_l) \frac{t_{n-k-1}^\alpha - t_{n-k}^\alpha}{\Delta t} , \]  

(27)

with \( \tilde{t}_k \in [t_k, t_{k+1}] \). From this, we consider the numerical scheme

\[ \psi^n_l = \psi^0_l - \frac{\Delta t \gamma^0 \gamma^1}{2 \Gamma(\alpha + 1)} \sum_{k=0}^{n-1} \left( \frac{(\psi^1_{x})^k+1_1 + (\psi^1_{x})^k_k}{2 \Delta x} \right) \frac{t_{n-k-1}^\alpha - t_{n-k}^\alpha}{\Delta t} . \]  

(28)
The local truncation error can be written as

\[ \mathcal{E}_l^n = E_n + \frac{\Delta x^2 \gamma^1}{6\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} \psi_{xxx}(\tilde{t}_k, \tilde{x}_l) \left( t_{n-k-1}^\alpha - t_{n-k}^\alpha \right) \]

\[ = E_n + \frac{\Delta x^2 \gamma^1}{6\Gamma(\alpha + 1)} \psi_{xxx}(\tilde{z}_n, \tilde{x}_l) \sum_{k=0}^{n-1} \left( t_{n-k-1}^\alpha - t_{n-k}^\alpha \right) \]

\[ = E_n - \frac{\Delta x^2 \gamma^1}{6\Gamma(\alpha + 1)} \psi_{xxx}(\tilde{z}_n, \tilde{x}_l) t_{n}^\alpha, \quad (29) \]

for some \( \tilde{z}_n \in [0, t_n] \). From this and what was observed for \( E_n \) in the previous section, we have that the scheme is locally of order 2 in both mesh parameters. This is an implicit scheme since the equation couples three values of \( \psi \) at the highest time-level \( n \). In order to show this clearly, we may rewrite (28) as:

\[ \psi_l^n = \psi_l^0 - \frac{\Delta t \gamma^0 \gamma^1}{2\Gamma(\alpha + 1)} \left( -\frac{\Delta t}{2 \Delta x} \psi_{l+1}^n - \psi_{l-1}^n + \frac{n-2}{2 \Delta x} \sum_{k=0}^{n-2} \psi_{l+1}^{k+1} - \psi_{l-1}^{k+1} t_{n-k-1}^\alpha - t_{n-k}^\alpha \frac{\Delta t}{\Delta t} \right) \]

\[ - \frac{\Delta t \gamma^0 \gamma^1}{2\Gamma(\alpha + 1)} \sum_{k=0}^{n-1} \frac{\psi_{l+1}^k - \psi_{l-1}^k}{2 \Delta x} t_{n-k-1}^\alpha - t_{n-k}^\alpha \frac{\Delta t}{\Delta t} \]

\[ \quad \iff \]

\[ \psi_l^n - \frac{\Delta t}{\beta} A \left( \psi_{l+1}^n - \psi_{l-1}^n \right) = \frac{\Delta t}{\beta} A \sum_{k=1}^{n-1} T_{n-k} \left( \psi_{l+1}^k - \psi_{l-1}^k \right) \]

\[ + \psi_l^0 + \frac{\Delta t}{\beta} A T_n^* \left( \psi_{l+1}^0 - \psi_{l-1}^0 \right), \quad (31) \]

with:

\[ \beta = 4\Gamma(\alpha + 1) \Delta x, \quad A = \gamma^0 \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

\[ T_{n-k} = \frac{t_{n-k+1}^\alpha - t_{n-k}^\alpha}{\Delta t}, \quad T_n^* = \frac{t_n^\alpha - t_{n-1}^\alpha}{\Delta t}. \quad (32) \]

### 2.2 Numerical analysis

We define the error at position \((n, l)\) as the difference between the exact solution \(\psi(t_n, x_l)\) and the numerical solution given by the scheme \(\psi_l^n\):

\[ e_l^n \equiv \psi(t_n, x_l) - \psi_l^n. \quad (33) \]

Since (31) is a linear expression, we have

\[ e_l^n - \frac{\Delta t}{\beta} A \left( e_{l+1}^n - e_{l-1}^n \right) = \frac{\Delta t}{\beta} A \sum_{k=1}^{n-1} T_{n-k} \left( e_{l+1}^k - e_{l-1}^k \right) \]

\[ + e_l^0 + \frac{\Delta t}{\beta} A T_n^* \left( e_{l+1}^0 - e_{l-1}^0 \right) + \mathcal{E}_l^n. \quad (34) \]
At this point, we must fix the boundary conditions. Since the whole analysis depends on this, a choice must be made. We have considered the simplest case, which corresponds to null boundary conditions on the solution. We consider a spatial region of finite length $X$, corresponding to $l$ ranging from 0 to $L + 1$, such that $X = \Delta x(L + 1)$, and assume that both the exact and the numerical solutions are zero for all times at the spatial boundaries. The numerical computations are done, thus, for $l = 1, \ldots, L$.

We stack for each time-step $n$ the errors in an $L$ dimensional error block-vector $\vec{e}^n$ such that its $l$ component is the $2 \times 1$ block $e^n_l$. Similarly, we construct the vector truncation error $\vec{E}^n$. (We must keep in mind that since $\psi$ is a two-component spinor, $e^n_l$ and $E^n_l$ have also two components.)

Due to our choice of the boundary conditions, we may write the error equation as

$$
\left( I - \frac{\Delta t^n}{\beta} P \right) \vec{e}^n = \frac{\Delta t^n}{\beta} \sum_{k=1}^{n-1} T_{n-k} P \vec{e}^k + \left( I + \frac{\Delta t^n}{\beta} T^n P \right) \vec{e}^0 + \vec{E}^n,
$$

with $I$ the $2L \times 2L$ identity matrix and $P$ the banded $2L \times 2L$ matrix given by

$$
P = M \otimes A, \quad M = \begin{pmatrix}
0 & 1 & 0 & \ldots & \ldots & 0 \\
-1 & 0 & 1 & \ddots & \ddots & 0 \\
0 & -1 & 0 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 1 & \ddots \\
0 & \cdots & \cdots & -1 & 0 & 1 \\
0 & \cdots & \cdots & 1 & \cdots & \cdots \\
0 & \cdots & \cdots & 0 & -1 & 0 \\
\end{pmatrix},
$$

where $\otimes$ is the Kronecker product and $M$ is $L \times L$. In $P$, only the block diagonals adjacent to the principal one are not null.

In the study of the convergence, we may consider that the initial error $\vec{e}^0$ is null, and we have from (35)

$$
Q\vec{e}^n = \delta P \sum_{k=1}^{n-1} T_{n-k} \vec{e}^k + \vec{E}^n,
$$

where we have defined

$$
\delta \equiv \frac{\Delta t^n}{\beta}, \quad Q \equiv I - \frac{\Delta t^n}{\beta} P;
$$

and the recurrence:

$$
\begin{align*}
Q\vec{e}^1 &= \vec{E}^1, \\
Q\vec{e}^2 &= \delta P Q^{-1} T_1 \vec{E}^1 + \vec{E}^2, \\
Q\vec{e}^3 &= (\delta P Q^{-1})^2 T_1^2 \vec{E}^1 + \delta P Q^{-1} \left( T_2 \vec{E}^1 + T_1 \vec{E}^2 \right) + \vec{E}^3, \\
Q\vec{e}^4 &= (\delta P Q^{-1})^3 T_1^3 \vec{E}^1 + (\delta P Q^{-1})^2 \left( 2 T_2 T_1 \vec{E}^1 + T_1^2 \vec{E}^2 \right) + (\delta P Q^{-1}) \left( T_3 \vec{E}^1 + T_2 \vec{E}^2 + T_1 \vec{E}^3 \right) + \vec{E}^4,
\end{align*}
$$

etc.
All scalar coefficients in (37) are positive and the $T$’s also satisfy:

$$j < k \implies T_j > T_k. \quad (40)$$

Taking an appropriate norm and defining $\| \vec{E} \|$ as the maximum over $n$ of $\| \vec{E}^n \|$, it is possible to show that the recurrence gives rise to

$$\forall n \geq 1, \quad \| \vec{E}^n \| \leq \| Q^{-1} \left( 1 + \delta T \| PQ^{-1} \| \right)^{n-1} \| \vec{E} \|, \quad (41)$$

In order to ensure convergence we need that the right-hand side of (41) goes to zero as $\Delta t$ and $\Delta x$ go to zero, with $n\Delta t = t_n$ and $(L + 1)\Delta x = X$ remaining constant.

In order to estimate $\| PQ^{-1} \|$ we will use the spectral norm. We will suppose in what follows that $L$ is even, so that $P$ is nonsingular. Since

$$QP^{-1} = P^{-1} - \frac{\Delta t^\alpha}{\beta} I, \quad (42)$$

if we denote by $\lambda$ the eigenvalues of $QP^{-1}$, we have:

$$|QP^{-1} - \lambda I| = 0 \iff \left| P^{-1} - \left( \frac{\Delta t^\alpha}{\beta} + \lambda \right) I \right| = 0$$

$$\iff \left( \frac{\Delta t^\alpha}{\beta} + \lambda \right) \text{ eigenvalue of } P^{-1}. \quad (43)$$

Let us call $\mu$ the eigenvalues of $P$, we have, thus,

$$\lambda = \frac{1}{\mu} - \frac{\Delta t^\alpha}{\beta}. \quad (44)$$

The eigenvalues of $P$ are simple to find: due to the Kronecker product (36), they are all the possible products between the eigenvalues of $M$ and those of $A$ (see [4], for instance). Due to the special diagonal form of $M$, its eigenvalues are given as a particular case of the formula for tridiagonal matrices (see [5], for instance), by

$$\mu_j = 2i \cos \left( \frac{j\pi}{L + 1} \right), \quad j = 1, \ldots, L; \quad (45)$$

on the other hand, the eigenvalues of $A$ are $1$ and $-1$. We have then, that all the eigenvalues of $P$ appear in conjugate pairs of pure imaginary values. This supposes that the corresponding $\lambda(\pm \mu_j)$ are complex eigenvalues, with modulus

$$|\lambda(\pm \mu_j)| = \sqrt{\frac{1}{4 \cos^2 \left( \frac{j\pi}{L + 1} \right)} + \frac{\Delta t^{2\alpha}}{\beta^2}}, \quad j = 1, \ldots, L. \quad (46)$$

Since the eigenvalues of $QP^{-1}$ and those of $PQ^{-1}$ are reciprocal, we have

$$\| PQ^{-1} \| \leq \min_j |\lambda(\mu_j)| = \left( \frac{1}{4 \cos^2 \left( \frac{L\pi}{L + 1} \right)} + \frac{\Delta t^{2\alpha}}{\beta^2} \right)^{-1/2}$$

$$\leq \left( \frac{4L^2}{4\pi^2} + \frac{\Delta t^{2\alpha}}{\beta^2} \right)^{-1/2} = \left( \frac{X'^2}{\Delta x^2\pi^2} + \frac{\Delta t^{2\alpha}}{\beta^2} \right)^{-1/2}$$

$$= \frac{4\pi\Delta x \Gamma(\alpha + 1)}{\sqrt{16\Gamma^2(\alpha + 1)X'^2 + \pi^2\Delta t^{2\alpha}}} \equiv K\Delta x. \quad (47)$$
We proceed in a similar way to compute \( \|Q^{-1}\| \): if we represent now by \( \lambda \) the eigenvalues of \( Q \), we have
\[
\left| I - \frac{\Delta t^\alpha}{\beta} P - \lambda I \right| = 0 \iff \frac{\beta}{\Delta t^\alpha} (1 - \lambda) \text{ eigenvalue of } P, \tag{48}
\]
and, thus,
\[
\lambda(\pm \mu_j) = 1 \mp \frac{\Delta t^\alpha}{\beta} \mu_j = 1 \mp 2i \frac{\Delta t^\alpha}{\beta} \cos \left( \frac{j\pi}{L + 1} \right), \quad j = 1, \ldots, L. \tag{49}
\]
We finally have
\[
\|Q^{-1}\| \leq \frac{1}{\min_j |\lambda(\mu_j)|} = \frac{1}{\min_j \sqrt{1 + 4 \left( \frac{\Delta t^{2\alpha}}{\beta^2} \cos^2 \left( \frac{j\pi}{L + 1} \right) \right)}} \leq 1. \tag{50}
\]
With all this, we have from (41):
\[
\forall n \geq 1, \quad \|\vec{e}^n\| \leq \left( 1 + \delta T_1 \|PQ^{-1}\| \right)^{n-1} \|\vec{E}\|
\leq \left( 1 + \frac{\Delta t}{4\Gamma(\alpha + 1) T_1 K} \right)^{n-1} \|\vec{E}\|
\leq \exp \left( \frac{(n - 1)\Delta t}{4\Gamma(\alpha + 1)} T_1 K \right) \|\vec{E}\|
\leq \exp \left( \frac{t_{n-1}}{4\Gamma(\alpha + 1)} T_1 K \right) \|\vec{E}\|. \tag{51}
\]
Since \( t_{n-1}, T_1 \) and \( K \) are bounded as \( \Delta t \) and \( \Delta x \) go to zero, and since \( \|\vec{E}\| \) is \( O(\Delta t^2 + \Delta x^2) \), the error bound goes to zero in the limit and the scheme is convergent if the initial error is zero and the boundary conditions are null. For the study of the stability, keeping the same boundary conditions, for instance, we should supppose a round-off term on the right-handside of (35), and \( \vec{e}^0 \) not null. This will give extra terms in (39) that should get properly bounded. The study of the stability is beyond the scope of this present work.

3 Conclusion

We have build a second order numerical scheme to simulate a fractional Dirac equation and have performed the numerical analysis of the convergence under reasonable assumptions.

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References


