On t-norms based measures of specificity

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Abstract

This paper gives a general expression for families of measures of specificity of a fuzzy set or a possibility distribution based on three t-norms and a negation. Other known measures of specificity are particular cases of this expression and new examples are provided. \copyright{} 2002 Published by Elsevier Science B.V.

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1. Introduction

The concept of specificity provides a measure of the amount of information contained in a fuzzy set or possibility distribution by giving a degree for a fuzzy set to contain just one element. It is strongly related to the inverse of the cardinality of a set.

Let us remember that:

- Specificity measures were introduced by Yager [14, 15, 16, 18, 20, 21, 22, 23, 24] showing its usefulness as a measure of tranquility when making a decision. Yager introduced the specificity-correctness trade-off principle. The output information of expert systems and other knowledge-based systems should be both specific and correct to be useful. Yager suggested the use of specificity in default reasoning, in possibility qualified statements and data mining processes, giving several possible manifestations of this measure.
- Dubois and Prade [4, 3] introduced the minimal specificity principle and showed the role of specificity in the theory of approximate reasoning.

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• Higashi and Klir [7] introduced a closely related idea called **non-specificity**.
• The concept of granularity introduced by Zadeh [29] is correlated with the concept of specificity.

This paper proposes a new general definition to express the concept of specificity by using three t-norms and a negation. It is shown that other known formulas are particular cases of this general definition and new measures of specificity potentially useful in many applications are provided.

### 2. Preliminaries

**Definition 1.** A fuzzy set $\mu$ on $X$ is **normal** if there exists an element $x_1 \in X$ such that $\mu(x_1) = a_1 = 1$.

**Definition 2** (Measure of specificity). Let $X$ be a set with elements $\{x_i\}$ and let $[0,1]^X$ be the class of fuzzy sets of $X$. A measure of specificity $Sp$ is a function $Sp : [0,1]^X \rightarrow [0,1]$ such that

1. $Sp(\mu) = 1$ if and only if $\mu$ is a singleton ($\mu = \{x_1\}$).
2. $Sp(\emptyset) = 0$.
3. If $\mu$ and $\eta$ are normal fuzzy sets in $X$ and $\mu \subset \eta$, then $Sp(\mu) \geq Sp(\eta)$.

The first condition imposes that the specificity is one (maximum value) only for crisp sets with just one element (singletons). The second condition assumes the minimum specificity for the null set. Other non-null fuzzy sets could also have specificity zero. The third condition requires that the specificity measure of a normal fuzzy set decreases when the membership degree of its elements increases.

If we would have to choose one element of a set of elements, and we have a fuzzy set with the degree of usefulness of each element, it is desirable to have a singleton or a high-specificity fuzzy set to be sure that our election is right.

**Definition 3** (Weak measure of specificity). Let $X$ be a set with elements $\{x_i\}$ and let $[0,1]^X$ be the class of fuzzy sets of $X$. A weak measure of specificity $Sp$ is a function $Sp : [0,1]^X \rightarrow [0,1]$ such that

1. $Sp(\mu) = 1$ if $\mu$ is a singleton ($\mu = \{x_1\}$).
2. $Sp(\emptyset) = 0$.
3. If $\mu$ and $\eta$ are normal fuzzy sets in $X$ and $\mu \subset \eta$, then $Sp(\mu) \geq Sp(\eta)$.

Note that the difference between a measure of specificity and a weak measure of specificity lies on axiom 1.

Specificity measures are not fuzzy measures [6] because they are not monotonous. The following definition of weak measure justifies the word ‘measures’, because specificity measures are weak measures.
Definition 4 (A weak concept of measure, Trillas and Alsina [13]). A measure of a characteristic \( k \) shown by the elements of a set \( E \) is made through a comparative relation like ‘\( x \) shows the characteristic \( k \) less than \( y \) shows it’ for any \( x, y \) in \( E \).

Let us write ‘\( x_k \leq y \)’ to denote that relation and suppose that \( \leq_k \) is a preorder on \( E \).

A function \( m : E \rightarrow [0, 1] \) is a \( \leq_k \)-measure for \( E \) whenever:

1. \( m(x_0) = 0 \) if \( x_0 \in E \) is minimal for \( \leq_k \).
2. \( m(x_1) = 1 \) if \( x_1 \in E \) is maximal for \( \leq_k \).
3. If \( x_k \leq y \) then \( m(x) \leq m(y) \).

Remarks. 1. Of course, fuzzy measures [6] are \( \subseteq \)-measures (monotonous measures), and fuzzy entropies [2] are \( \leq_S \)-measures, where \( \subseteq \) is the contention and \( \leq_S \) is the sharpened ordering.

2. Weak measures of specificity effectively measures the idea of how close is a fuzzy set from a singleton. So, a measure of specificity \( Sp \) is a weak measure where the set \( E \) is \([0, 1]^X\), the characteristic \( k \) is the specificity of a fuzzy set, \( x_0 \) is the empty set, \( x_1 \) is a singleton and the preorder \( \leq_{Sp} \) is defined as \( \mu \leq_{Sp} \sigma \iff Sp(\mu) \leq Sp(\sigma) \).

3. Using the associative property of t-norms and t-conorms, generalized \( n \)-argument t-norms and t-conorms are easily defined [1].

3. t-norms and negation-based weak measure of specificity

Definition 5 (Measure of \( T \)-specificity \( Sp_T \)). Let \( \mu \) be a fuzzy set in a finite set \( X \), and let \( a_i \) be the membership degree of the element \( x_i \) (\( \mu(x_i) = a_i \)). The membership degrees \( a_i \in [0, 1] \) are totally ordered with \( a_1 \geq a_2 \geq \cdots \geq a_n \). Let \( N \) be a negation [12], let \( T_1 \) and \( T_3 \) be any t-norms and let \( T_2^* \) an \( n \)-argument t-conorm. Let \( T \) be the quartet \((T_1, N, T_2, T_3)\). Let \( \{w_j\} \) be a weighting vector.

A measure of \( T \)-specificity \( Sp_T \) is an application \( Sp_T : [0, 1]^X \rightarrow [0, 1] \) defined by

\[
Sp_T(\mu) = T_1(a_1, N(T_2^*_{j=2} \cdots n(T_3(a_j, w_j)))).
\]

Note: This formula represents the logical idea of ‘one element’ (represented by its membership degree \( a_1 \)) ‘and no others’. This first ‘and’ is implemented through the t-norm \( T_1 \). The negation of other elements is represented by a negation \( N \) of a general \( n \)-argument t-conorm \( T_2^* \) and the t-norm \( T_3 \).

Notation. Let us denote by \( F(\mu) \) the function \( T_2^*_{j=2} \cdots n(T_3(a_j, w_j)) \), so

\[
Sp_T(\mu) = T_1(a_1, N(F(\mu))) = T_1(a_1, N(T_2^*_{j=2} \cdots n(T_3(a_j, w_j)))) = T_1(a_1, N(F(\mu))) = T_1(a_1, N(T_2^*_{j=2} \cdots n(T_3(a_j, w_j)))) = T_1(a_1, N(F(\mu))).
\]

The three following lemmas prove that measures of \( T \)-specificity are weak measures of specificity.

Lemma 1. If \( \mu \) is a singleton then \( Sp_T(\mu) = 1 \).
Proof. Let $\mu$ be a singleton, then $a_1 = 1$ and $a_j = 0$ for all $j$, $2, \ldots, n$. So
\[
\text{Sp}_T(\mu) = T_1(a_1, N(F(\mu))) = T_1(1, N(F(\mu))) = N(F(\mu)) = N(T_2^*_{j=2,\ldots,n}\{T_3(0, w_j)\})
\]
\[
= N(T_2^*_{j=2,\ldots,n}\{0, \ldots, 0\}) = N(0) = 1.
\]
\[\Box\]

Lemma 2. The $T$-specificity of the empty set is zero ($\text{Sp}_T(\emptyset) = 0$).

Proof. $a_j = 0$ for all $j$, so $\text{Sp}_T(\mu) = T(a_1, N(F(\mu))) = T(0, N(F(\mu))) = 0$. \[\Box\]

Lemma 3. If $\mu$ and $\eta$ are normal fuzzy sets in $X$ and $\mu \subset \eta$, then $\text{Sp}_T(\mu) \geq \text{Sp}_T(\eta)$.

Proof. Let $a_j$ and $b_j$, respectively, be the $j$th greatest membership degree of $\mu$ and $\eta$. $\mu \subset \eta$ so $a_j \leq b_j$ for all $j$ and $T_2^*_{j=2,\ldots,n}\{T_3(a_j, w_j)\} \leq T_2^*_{j=2,\ldots,n}\{T_3(b_j, w_j)\}$. $\mu$ and $\eta$ are normal, so $a_1 = b_1 = 1$, and
\[
\text{Sp}_T(\mu) = T_1(a_1, N(F(\mu))) = T_1(1, N(F(\mu))) = N(F(\mu)) = N(T_2^*_{j=2,\ldots,n}\{T_3(a_j, w_j)\})
\]
\[
\geq N(T_2^*_{j=2,\ldots,n}\{T_3(b_j, w_j)\}) = T_1(1, N(F(\eta))) = T_1(a_1, N(F(\eta))) = \text{Sp}_T(\eta).
\]
\[\Box\]

Theorem 1. A measure of $T$-specificity is a weak measure of specificity.

Proof. The proof follows from Lemmas 1–3. \[\Box\]

Definition 6. A t-norm $T$ is positive [10] when $T(x, y) = 0$ if and only if $x = 0$ or $y = 0$.

For example, the minimum t-norm and all t-norms in the family of the product t-norm are positive t-norms.

Lemma 4. If $T_3$ is a positive t-norm, $N$ is a strong negation and the weight $w_2$ is greater than zero, then the measure of $T$-specificity is a measure of specificity.

Proof. Theorem 1 shows that a measure of $T$-specificity is a weak measure of specificity. It is proven that if $\text{Sp}_T(\mu) = 1$ then $\mu$ is a singleton.

$N$ is a strong negation, so $N(x) = 1$ if and only if $x = 0$.

Suppose that $\mu$ is not a singleton.

Case 1: $a_2 = 0$. Then $a_j = 0$ for all $j$, $2, \ldots, n$, and $N(F(\mu)) = N(0) = 1$. So $\text{Sp}_T(\mu) = T_1(a_1, 1) = a_1$. $\mu$ is not a singleton so $\text{Sp}_T(\mu) = a_1 \neq 1$.

Case 2: $a_2 \neq 0$. Then $T_3(a_2, w_2) > 0$, so $T_2^*_{j=2,\ldots,n}\{T_3(a_j, w_j)\} > 0$, $N(T_2^*_{j=2,\ldots,n}\{T_3(a_j, w_j)\}) < 1$ and $\text{Sp}_T(\mu) = T_1(a_1, N(T_2^*_{j=2,\ldots,n}\{T_3(a_j, w_j)\})) < 1$. So if $\mu$ is not a singleton then $\text{Sp}_T(\mu) \neq 1$. \[\Box\]
Definition 7. A weak measure of specificity is lower strict when $\text{Sp}(\mu) = 0$ if and only if $\mu$ is the null set.

Lemma 5. If both t-norms $T_1$ and $T_2$ are positive, $N$ is a strong negation and $w_j < 1$ for all $j: 2, \ldots, n$ then the T-specificity measure is lower strict.

Proof. Lemma 2 shows that if $\mu$ is the null set then $\text{Sp}_T(\mu) = 0$. It is proven that if $\text{Sp}_T(\mu) = 0$ then $\mu$ is the null set.

$T_2$ is positive, so the dual t-conorm $T_2^*(x_1, \ldots, x_n) = 1$ if and only if exists $j$ such that $x_j = 1$. But $w_j < 1$, so $T_3(a_j, w_j) \leq w_j < 1$ for all $j: 2, \ldots, n$. Thus $T_2^*_{j=2, \ldots, n}\{T_3(a_j, w_j)\} < 1$. $N$ is a strong negation so $N(T_2^*_{j=2, \ldots, n}\{T_3(a_j, w_j)\}) = N(F(\mu)) > 0$.

Suppose that $\mu$ is not the null set, so $a_1 > 0$ and $\text{Sp}_T(\mu) = T_1(a_1, N(F(\mu))) > 0$. □

Corollary 1. If $\mu$ and $\eta$ are not null crisp subsets of $X$ and $\text{card}(\mu) \geq \text{card}(\eta)$ then $\text{Sp}_T(\mu) \leq \text{Sp}_T(\eta)$.

Proof. $\mu$ and $\eta$ are crisp sets such that $a_j = 1$ for $j: 1, \ldots, m$ ($m = \text{card}(\mu)$) and $a_j = 0$ for $j: m+1, \ldots, n, b_j = 1$ for $j: 1, \ldots, s$ ($s = \text{card}(\eta)$) and $b_j = 0$ for $j: s+1, \ldots, n, m \geq s$.

$$\text{Sp}_T(\mu) = T_1(a_1, N(F(\mu))) = T_1(1, N(F(\mu))) = N(F(\mu)) = N(T_2^*_{j=2, \ldots, n}\{T_3(a_j, w_j)\})$$

$$= N(T_2^*_{j=2, \ldots, n}\{T_3(1, w_2), \ldots, T_3(1, w_m), T_3(0, w_{m+1}), \ldots, T_3(0, w_d)\})$$

$$= N(T_2^*_{j=2, \ldots, n}\{w_2, \ldots, w_m, 0, \ldots, 0\})$$

$$\leq N(T_2^*_{j=2, \ldots, n}\{w_2, \ldots, w_s, 0, \ldots, 0\}) = \text{Sp}_T(\eta).$$ □

Lemma 6. If $\mu$ is a crisp set with cardinal $m$, $1 < m \leq n$, the greatest weight is $w_M$ and $T_2^*$ is the $n$-argument t-conorm maximum then $\text{Sp}_T(\mu) = N(w_M)$.

Proof. $\mu$ is a crisp set such that $a_j = 1$ for $j: 1, \ldots, m$ ($m = \text{card}(\mu)$) and $a_j = 0$ for $j: m+1, \ldots, n$.

$$\text{Sp}_T(\mu) = T_1(a_1, N(F(\mu))) = T_1(1, N(F(\mu))) = N(F(\mu)) = N(T_2^*_{j=2, \ldots, n}\{T_3(a_j, w_j)\})$$

$$= N(\text{Max}\{T_3(1, w_2), \ldots, T_3(1, w_m), T_3(0, w_{m+1}), \ldots, T_3(0, w_n)\})$$

$$= N(\text{Max}\{w_2, \ldots, w_m, 0, \ldots, 0\}) = N(\text{Max}\{w_2, \ldots, w_m\} = N(w_M)).$$ □

Theorem 2. If $L = (\land, \lor, ^\prime)$ is a logic triplet based on a t-norm $\land$, its dual t-conorm $\lor$ and a negation $^\prime$ then

$$\text{Sp}_L(\mu) = a_1 \land (a_2^\prime \lor w_2^\prime) \land \cdots \land (a_n^\prime \lor w_n^\prime) \text{ is a weak measure of specificity.}$$
Proof. It is shown that $Sp_L$ is a particular case of measure of $T$-specificity with $T=(\wedge, ', \wedge, \wedge)$ which are weak measures of specificity. Suppose that $T_1 = T_2 = T_3 = \wedge$, and that $N = '$.

$$Sp_T(\mu) = T_1(a_1, N(T_2^* = \wedge, n \{T_3(a_j, w_j)\})) = T_1(a_1, T_2^* = \wedge, n \{N(T_3(a_j, w_j))\})$$
$$= a_1 \wedge (a_2' \vee w_2') \wedge \cdots \wedge (a_n' \vee w_n') = Sp_L(\mu). \quad \square$$

This expression allows a new interpretation of weak specificity measures as $a_1 \wedge P_2 \wedge \cdots \wedge P_n$ where values $P_j$ are penalties for elements $x_2, \ldots, x_n$.

Corollary 2. Let $L = (\wedge, \vee, ')$ be a logical triplet based on a positive $t$-norm $\wedge$, its dual $t$-conorm $\vee$ and a negation $'$. If $w_2 > 0$ then

$$Sp_L(\mu) = a_1 \wedge (a_2' \vee w_2') \wedge \cdots \wedge (a_n' \vee w_n')$$

is a measure of specificity.

Proof. The proof follows from Theorem 2 and Lemma 4. \square

Definition 8. Let $Sp$ and $Sp^*$ be measures of specificity on the space $X$. $Sp$ is more critical than $Sp^*$ when their respective weights $w_j$ and $w_j^*$ verify $w_j \geq w_j^*$ for all $j$.

Definition 9. Let $Sp$ and $Sp^*$ be measures of specificity on the space $X$. $Sp$ is stricter than $Sp^*$, denoted by $Sp \leq Sp^*$ [26], if for all fuzzy subsets $\mu$ of $X$ $Sp(\mu) \leq Sp^*(\mu)$.

Definition 10. The $T$-class of weak measures of specificity is the set of measures $Sp_T$ defined by the same $t$-norms and the same negation.

Lemma 7. Let $Sp_T$ and $Sp_T^*$ be weak measures of specificity in the same $T$-class of weak measures of specificity. If $Sp_T$ is more critical than $Sp_T^*$ then $Sp_T$ is stricter than $Sp_T^*$.

Proof. $Sp_T$ is more critical than $Sp_T^*$, so $w_j \geq w_j^*$ for all $j$. Thus $T_3(a_j, w_j) \geq T_3(a_j, w_j^*)$ for all $j$ and $F(\mu) \geq F^*(\mu)$. So $N(F(\mu)) \leq N(F^*(\mu))$ and $Sp_T(\mu) \leq Sp_T^*(\mu). \quad \square$

Definition 11. A weak measure of specificity is regular [26] if for all constant fuzzy sets $(\mu_c(x) = c$ for all $x$) $Sp_T(\mu_c) = 0$.

4. Examples

Measures of $T$-specificity allow to obtain many different expressions of weak measures of specificity and measures of specificity of a fuzzy set or a possibility distribution in order to evaluate the usefulness of the information in many different environments. Measures of $T$-specificity provide a
simple general formula that could be useful to implement any measure of weak specificity needed in applications.

It is shown that the most important known measures of specificity for a finite space are measures of $T$-specificity.

**Example 1.** Yager introduced [20] the linear measures of specificity on a finite space $X$ as

$$Sp(\mu) = a_1 - \sum_{j=2}^{n} w_j a_j,$$

where $a_j$ is the $j$th greatest membership degree of $\mu$ and $\{w_j\}$ is a set of weights verifying:

1. $w_j \in [0, 1]$.
2. $\sum_{j=2}^{n} w_j = 1$.
3. $w_j \geq w_i$ for all $1 < j < i$.

**Theorem 3.** Linear measures of specificity are measures of $T$-specificity with $T = (W, N, W, Product)$ where $W$ is the Lukasiewicz t-norm and $N$ is the negation $N(x) = 1 - x$.

**Proof.** Let $T_1$ and $T_2$ be the Lukasiewicz t-norm defined by $T_1(a, b) = \max\{0, a + b - 1\}$ and $T_2^*(a_1, \ldots, a_n) = \min\{1, a_1 + \cdots + a_n\}$.

$$Sp_T(\mu) = T_1(a_1, N(T_2^*_{j=2,...,n}\{T_3(a_j, w_j)\})) = \max\{0, a_1 + N(F(\mu)) - 1\}$$

$$= \max\{0, a_1 + (1 - F(\mu)) - 1\} = \max\{0, a_1 - F(\mu)\}$$

$$= \max\{0, a_1 - T_2^*_{j=2,...,n}\{T_3(a_j, w_j)\}\}$$

$$= \max \left\{0, a_1 - \min \left\{1, \sum_{j=2}^{n} w_j a_j\right\}\right\}$$

$$= a_1 - \sum_{j=2}^{n} w_j a_j. \quad (1)$$

It follows the explanation of the last equality:

(1) $a_j \leq 1 \Rightarrow \sum_{j=2}^{n} w_j a_j \leq \sum_{j=2}^{n} w_j 1 = \sum_{j=2}^{n} w_j = 1 \Rightarrow \min\{1, \sum_{j=2}^{n} w_j a_j\} = \sum_{j=2}^{n} w_j a_j.$

(2) $a_j \geq a_j \Rightarrow \sum_{j=2}^{n} w_j a_j \leq \sum_{j=2}^{n} w_j a_1 = a_1 \sum_{j=2}^{n} w_j = a_1 \Rightarrow a_1 - \sum_{j=2}^{n} w_j a_j \geq 0 \Rightarrow \max\{0, a_1 - \sum_{j=2}^{n} w_j a_j\} = a_1 - \sum_{j=2}^{n} w_j a_j. \quad (2)$

**Lemma 8.** Linear measures of specificity are measures of specificity.

**Proof.** It follows from Theorem 3 that linear measures of specificity are measures of $T$-specificity and, from Theorem 1, they are also weak measures of specificity.
In the case of the linear $T$-measure of specificity, $T_3$ is the product, $N(x) = 1 - x$ is a strong negation and conditions 2 and 3 of the weights for linear measures of specificity imply that $w_2 > 0$, so the proof follows from Lemma 4.

**Properties (Yager [26]).**

- Linear measures of specificity are regular.
- $\text{Sp}(\mu) = a_1 - a_2$ is the strictest linear measure of specificity.
- The less stricter linear measure of specificity is

$$\text{Sp}(\mu) = a_1 - \frac{1}{n-1} \sum_{j=2}^{n} a_j.$$ 

**Corollary 3.** Yager’s measure of specificity [26] on a finite space $X$ defined by

$$\text{Sp}(\mu) = \int_{0}^{x_{\text{max}}} \frac{1}{\text{Card}(\mu_x)} \, dx,$$

is a measure of $T$-specificity.

**Proof.**

$$\int_{0}^{x_{\text{max}}} \frac{1}{\text{Card}(\mu_x)} \, dx$$

is a particular case of linear measure of specificity taking the weights as $w_2 = \frac{1}{2}$ and $w_j = 1/(j - 1) - 1/j$ for all $j > 2$, so following Theorem 3 it is also a measure of $T$-specificity with $T = (W, N, W,$ Product).

**Example 2.** Yager [26] defined the product measure of specificity for multi-criteria decision-making problems by $\text{Sp}(\mu) = a_1 \prod_{j=2}^{n} (ka_j + (1 - a_j))$, where $k \in [0, 1]$.

This formula measures the existence of an element with membership degree one and all others with membership degree zero.

**Theorem 4.** $\text{Sp}(\mu) = a_1 \prod_{j=2}^{n} (ka_j + (1 - a_j))$ where $k \in [0, 1]$ is a measure of $T$-specificity with $T = (\text{Prod}, N, \text{Prod}, \text{Prod})$ and $w_j = 1 - k$ for all $j$.

**Proof.** If $T = (\text{Prod}, N, \text{Prod}, \text{Prod})$ and $w_j = 1 - k$ for all $j$ then:

$$\text{Sp}_T(\mu) = T_1(a_1, N(T_2^{*} \{T_3(a_j, w_j)\}))) = T_1(a_1, T_2^{*} \{N(T_3(a_j, w_j))\})$$

$$= a_1 \prod_{j=2}^{n} N(a_j, w_j) = a_1 \prod_{j=2}^{n} 1 - a_j w_j = a_1 \prod_{j=2}^{n} 1 - (1 - k) a_j$$
Corollary 4. If $w_2 > 0$ then the product measure of specificity is a measure of specificity, and if $w_j < 1$ for all $j$ then the product measure of specificity is a lower strict measure of specificity.

Proof. It follows from Theorem 4 and Lemmas 4 and 5.

Example 3. A more general example of product measures of specificity in the same product-class of measures of specificity is

$$Sp(\mu) = a_1 \prod_{j=2}^{n} (1 - w_j a_j) \text{ where } w_j \in [0,1].$$

Corollary. If $w_2 > 0$ then the general product measure of specificity is a measure of specificity, and if $w_j < 1$ then the general product measure of specificity is lower strict.

Proof. It follows from Theorem 4 and Lemmas 4 and 5.

Example 4 (Distance related measures of specificity). Another point of view for measures of specificity are distance-related measures of specificity. A fuzzy set $\mu$ of a set $X$ with cardinal $n$ can be seen as a vector of dimension $n$ or as a point in $[0,1]^n$. Let $E_i$ be the characteristic function of the singleton $(0,\ldots,\mu(i),\ldots,0)$, which can be seen as a collection of base vectors. The distance-related measure of specificity of a fuzzy set is defined through a negation operation of the closest distance of the fuzzy set with a singleton.

Let $d_p$ be the $p$-euclidean distance defined by

$$d_p(\mu, \eta) = \left( \sum_{j=1}^{n} |a_j - b_j|^p \right)^{1-p}.$$  

Yager shows [26] that the normalized metric $F(d_p(\mu, \eta)) = \min(1, d_p(\mu, \eta))$ is also a $W$-distance, it is, a distance satisfying the $W$-triangular inequality ($F(d_p(\mu, \eta)) \leq W(F(d_p(\mu, \sigma)), F(d_p(\sigma, \eta)))$) for all $\mu, \sigma, \eta$ in $[0,1]^X$ and defines the measure of specificity of a fuzzy set $\mu$ as

$$S_p(\mu) = 1 - \min_{i} d(\mu, E_i))$$

Note:

$$W_p^*(x_1, \ldots, x_n) = \min \left( 1, \sqrt[p]{ \sum_{j=1}^{n} x_j^p } \right)$$

is a t-conorm in the family of Lukasiewicz t-conorms because $W_p^* = \varphi^{-1} \circ W \circ \varphi \times \varphi$ with $\varphi(x) = x^p$.  


Theorem 5. Euclidean $p$-distances related measures of specificity of normal fuzzy sets are measures of $T$-specificity with $T = (T_1, N, W^*_p, T_3)$ and $w_j = 1$ for all $j$, where $T_1$ and $T_3$ are any t-norms and $N$ is the negation $N(x) = 1 - x$.

Proof. $\mu$ is a normal fuzzy set, so $a_1 = 1$ and the closest singleton point is $E_1$ (see [26]). So

$$\text{Sp}_T(\mu) = T_1(a_1, N(W^*_p, \{T_3(a_j, w_j)\}))$$

$$= T_1(1, N(W^*_p, \{T_3(a_j, 1)\}))$$

$$= N(W^*_p, \{a_j\})$$

$$= 1 - \left( \min \left( 1, \sqrt[p]{\sum_{j=2}^{n} a_j^p} \right) \right)$$

$$= 1 - \left( \min(1, \sqrt[p]{|1 - 1| + |a_2^p - 0| + \cdots + |a_n^p - 0|}) \right)$$

$$= 1 - \left( \min(1, d_p(\mu, E_1)) \right)$$

$$= 1 - \min_i (d_p(\mu, E_i)). \quad \square$$

Theorem 6. Let $d_0$ be the distance defined by $d_0(\mu, \eta) = \max_{j=1,\ldots,n} (|a_j - b_j|)$. 0-distances related measures of specificity of normal fuzzy sets are measures of $T$-specificity with $T = (T_1, N, \text{Minimum}, T_3)$ and $w_j = 1$ for all $j$, where $T_1$ and $T_3$ are any t-norms and $N$ is the negation $N(x) = 1 - x$.

Proof. $\mu$ is a normal fuzzy set, so $a_1 = 1$ and the closest singleton point is $E_1$. So

$$\text{Sp}_T(\mu) = T_1 \left( a_1, N \left( \min_{j=2,\ldots,n} \{T_3(a_j, w_j)\} \right) \right)$$

$$= T_1 \left( 1, N \left( \max_{j=2,\ldots,n} \{T_3(a_j, 1)\} \right) \right)$$

$$= N \left( \max_{j=2,\ldots,n} \{a_j\} \right)$$

$$= 1 - \max_{j=2,\ldots,n} |a_j - 0|$$

$$= 1 - \max\{|1 - 1|, |a_2 - 0|, \ldots, |a_n - 0|\}$$

$$= 1 - \left( \min(1, d_0(\mu, E_1)) \right)$$

$$= 1 - \min_i (d_0(\mu, E_i)). \quad \square$$
5. Conclusion

Given three t-norms, a negation and a set of weights is defined as measure of \( T \)-specificity, which is proven to be a weak measure of specificity. The measure of \( T \)-specificity formula expresses the logical idea of ‘one element and no others’. The first t-norm \( T_1 \) represents this first ‘and’ and should not be the minimum t-norm in order to not lose information. This provides an easy way to build up weak measures of specificity and measures of specificity formulas that could be used in many different applications. Most used measures of specificity are shown to be a particular case of measures of \( T \)-specificity.

References


