# FRACTIONAL EQUATIONS WITH INTERNAL DEGREES OF FREEDOM


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**Symposium on Applied Fractional Calculus**

**Industrial Engineering School / University of Extremadura**

**Badajoz, October 15-17, 2007**

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**NONLOCALITY**:

*IN SPACE* : Long Range Interactions (Many Space Scales)

*IN TIME* : Effects with Memory / Delay (Many Time Scales)

***INTEGRODIFFERENTIAL // INTEGRAL EQUATIONS***

## Scenarios of Integral Equations

- **Potential Theory**: Newton's inverse square law of gravitational attraction and Coulomb's law in electromagnetism.
- **Problems in Geophysics**: Three dimensional map of Earth's interior. Gravimetric methods.
- **Problems in Electricity and Magnetism.**
- **Hereditary Phenomena in Physics (***materials with memory; hysteresis***) and Biology (***ecological processes: accumulation of metals).*
- **Problems in Population Growth and Industrial Replacement.**
- **Radiation Problems.**
- **Optimization, Automatic Control Systems.**
- **Communication Theory.**
- **Mathematical Economics.**

## *PHYSICAL CONTEXTS WITH THE SAME EQUATION*

| | ***DARCY*** LAW $q=-K$ ***Grad*** $h$ | ***FOURIER*** LAW $Q=-\kappa$ ***Grad*** $T$ | ***FICK*** LAW $f=-D$ ***Grad*** $C$ | ***OHM*** LAW $j=-\sigma$ ***Grad*** $V$ |
|---|---|---|---|---|
| ***Flux of*** | ***Groundwater*** $q$ | ***Heat:*** $Q$ | ***Solute:*** $f$ | ***Charge:*** $j$ |
| ***Potential*** | ***Head:*** $h$ | ***Temperature*** $T$ | ***Concentration*** $C$ | ***Voltage:*** $V$ |
| ***Medium Property*** | ***K: Hydraulic Conductivity*** | $\kappa$: ***Thermal Conductivity*** | ***D***: ***Diffusion Coefficient*** | $\sigma$: ***Electrical Conductivity*** |

## WAVES + FRACTALS → FRACTIONAL CALCULUS (1)

- XIX Century: James Clerk Maxwell and Lor Rayleigh studied the interaction of electromagnetic waves with Euclidean regular structures (cilinders, spheres,…).
- There are either nonregular artificial structures or from Nature that show many lenght scales and they are no suitable to be studied in the Euclidean context:
    - Nonregular surfaces, disordered media, structures with ***specific properties of scattering***,..etc.
    - Relation between the geometrical parameters (structure descriptors) and the physical quantities that characterize electromagnetically the system.
    - Tecnology: New space and time scales.

## WAVES + FRACTALS → FRACTIONAL CALCULUS (2)

- **Geometrical Optics:**
  - **Wave length $\lambda$<<<< Dimension of any change in the media. The eikonal is not longer valid.**

- **The Geometrical Optics cannot be applied in fractal media.**

- **Stationary eigenvalue problem:**
  - **Wave equation in a fractal potential.**
  - **Wave equation with fractal boundary conditions:**

**Ex. $Lu = \lambda u$**

**-L is a linear differential operator on $R^n$ with boundary conditions $u_0(x)$ on a non-differentiable surface but which admits the fractional derivative $D^\beta$ with $\beta < 1$.**

**- If we define $\Phi = D^{\beta-1} u$ , we have the problem $L \Phi = \lambda \Phi$ with the boundary condition $\Phi_0(x)$, being $\Phi$ differentiable**

***The new boundary problem is smooth!***

## Application: Distribution of Suspended Particles in the Atmosphere + Radiation Effects

- The family of fractional differential equations

$$D_t^{\alpha} u(t,x) - a\, D_x^{\beta} u(t,x) = 0 \quad , \quad t>0,\ x \in R,$$

associated to diffusion pocesses allows to define a set of probability distributions which are an analytic instrument to approximate the study of problems as particles suspended in the atmosphere, radiation,..etc

  - ***To characterize the influence on the radiation arriving to the Earth surface (dispersion + absorption → Optical Depth $\tau(\lambda)$ is a measure of the radiation damping)***

- Example: Junge Distribution $N(z,a) = C(z,a)\, a^{-(1+\nu)}$ where
  - ***$z$ is the high in the atmosphere; $a$ is the size particles (tipically for aerosols 0.01-10 μm ); and $2 < \nu < 4$***
  - ***$C(z,a)$ is a scale factor depending on the particle concentration.***
  - $\boldsymbol{\tau(\lambda) = k\, \lambda^{(2-\nu)}}$

# Mars Exploration

- ***REMS-MSL Project (Approved)***

  ***(Rover Environmental Monitoring Station – Mars Science Laboratory)***

  ***NASA Mission to Mars (2009, 2011?)***

  ***→ Models of the Boundary Layer and Martian Atmosphere***

  ***Pressure, Humidity, Temperature (Air and Ground), UV Radiation and Wind.***

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- ***MiniHUM Project (Approved)***

  ***ESA Mission to Mars (2011,2013?)***

  ***→Models of diffusion processes in the Martian Ground***

- ***METNET (Meteorological Network) Project***
  - ***Precursor: 2 Stations (2009, 2011?) (Approved)***
  - ***Global: 15 Stations (2015?) (Evaluation Process)***

# Basic Considerations (1)

- Fundamental Theorem of Calculus:
  - $dX/dt = F(t)$ , $X(0) = Xo$

$$\downarrow$$

$$X(t) = Xo + \int_o^t \; 1 \; F(\tau)\, d\tau$$

$$\downarrow$$

$$X(t) = Xo + \int_o^t K(t-\tau) \; F(\tau)\, d\tau$$

***Question:***

***Integral Transform* $\leftrightarrow$ *Fractional Derivative ?***

- Roots in the Complex Plane:

$$x^3 = 1 \rightarrow \; R_1, R_2, R_3$$

# Basic Considerations (2)

- Numerical Schemes for Systems of first and second order:
  - $dX/dt = F(X)$ , $X(0) = Xo$

    $\downarrow$
  - $d^2X/dt^2 = F(X)\, dX/dt = F(X)\ F'(X) = 1/2\ dF(X)^2/dX$

    $\downarrow$

***Newton Equation: $d^2X/dt^2 = G(X) = -\, dU(X)/dX$***

» ***$U(X)$ = Potential Energy $\rightarrow$ $U(X) = -1/2\ F(X)^2$***

» ***Conservative Schemes, Symplectic Schemes.***

# CONTINUOUS MEDIA THEORY: *TIMOSHENKO EQUATION-(1)*

$$\partial^4\varphi/ \partial x^4-(a^2 +b^2)\, \partial^4\varphi / \partial x^2\, \partial t^2 +a^2 b^2\, \partial^4\varphi/ \partial t^4+ a^2 c^2\, \partial^2\varphi/ \partial t^2=0$$

- **Flexural vibrations of an infinite uniform beam free from lateral loading and including the shear deflection of the beam:**
  - **1/a has the dimension of a velocity.**
  - **1/b has the dimension of a velocity and it is related to the shear modulus of elesticity.**
  - **c=1/R, R is the radius of gyration of the cross section.**

- **The Timoshenko equation was introduced to avoid the unphysical behaviour of the Rayleigh equation** $a^2 c^2\, \partial^2\varphi/ \partial t^2 + \partial^4\varphi/ \partial x^4 = 0$, **which is not accurate to describe the effect of impact loads on a beam: the phase and group velocities tend to infinity as the wave length tend to zero.**
  - » $\omega = k^2 /ac$

# CONTINUOUS MEDIA THEORY: *TIMOSHENKO EQUATION-(2)*

$$\partial^4\varphi/ \partial x^4-(a^2 +b^2)\, \partial^4\varphi / \partial x^2\, \partial t^2 +a^2 b^2\, \partial^4\varphi/ \partial t^4+ a^2 c^2\, \partial^2\varphi/ \partial t^2=0$$

- **If a=b the square root of Timoshenko equation has a simple algebraic structure:**
  - $i\, a\, c\, \partial\varphi/ \partial t = a^2\, \partial^2\varphi/ \partial t^2 - \partial^2\varphi/ \partial x^2$
- **We can name this equation:**

  ***Schrödinger—Klein-Gordon equation***
  - **The dispersion relation is:** $\omega = (k^2- a^2\, \omega^2) /ac$
- **Relativistic and nonrelativistic properties.**
- **If a≠b the algebraic structure is more complicated.**

# Fractional Diffusion Equation

$$A\frac{\partial^{\alpha}\Psi}{\partial t^{\alpha}} + B\frac{\partial\Psi}{\partial x} = 0$$

$$A\frac{\partial\Psi}{\partial t} + B\frac{\partial\Psi}{\partial x} = 0$$

$$\boxed{A\frac{\partial^{1/2}\Psi}{\partial t^{1/2}} + B\frac{\partial\Psi}{\partial x} = 0}$$

$$A^2 = I$$

$$B^2 = -I$$

$$\{A, B\} = 0$$

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$\gamma = 2\alpha$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^{\gamma} u}{\partial t^{\gamma}} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$A\frac{\partial^{1/2}\Psi}{\partial t^{1/2}} + B\frac{\partial\Psi}{\partial x} = 0 \qquad \Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

- We can interpret it as a system with two coupled diffusion processes or a diffusion process with internal degrees of freedom.
- The components φ and χ satisfy the classical diffusion equation and they are named *difunors* in analogy with the *spinors* of Quantum Mechanics.
- It is other panoramic view of the possible interpolations between the hyperbolic operator of the wave equation and the parabolic one of the classical diffusion equation.
- According to the representation of the Pauli algebra of A and B, we have either an uncoupled system or a coupled system of equations.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{cases} \partial_t^\alpha \varphi = \varphi \\ \partial_t^\alpha \chi = -\chi \end{cases}$$

$$A\frac{\partial^\alpha \Psi}{\partial t^\alpha} + B\frac{\partial \Psi}{\partial x} = 0 \qquad \overset{\gamma = 2\alpha}{\longleftrightarrow} \qquad \frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} = 0$$

***Time Inversion*** (t—>-t)

- If $\alpha$=1 we have the Dirac and wave equations which are invariant under time inversion.
- If $\alpha$=½ the classical diffusion equation and its square root *are not* invariants under time inversion.
- Interpolation for : 0< $\alpha < 1$. The invariance under time inversion is satisfied for
    - Dirac Fractional Equation:

$$\alpha = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots, \frac{3}{5}, \frac{3}{7}, \frac{3}{9}, \ldots, \frac{5}{7}, \frac{5}{9}, \frac{5}{11}, \ldots$$

    - Diffusion Fractional Equation:

$$\alpha = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}, \frac{1}{9}, \ldots,$$

$$A\frac{\partial^\alpha \Psi}{\partial t^\alpha} + B\frac{\partial \Psi}{\partial x} = 0 \qquad \overset{\gamma = 2\alpha}{\longleftrightarrow} \qquad \frac{\partial^\gamma u}{\partial t^\gamma} - \frac{\partial^2 u}{\partial x^2} = 0$$

***Space-Time Inversion*** (x—>-x, t—>-t)

- *Both equations are invariants under space inversion.*
- Interpolation : 0< α < 1. The invariance under space-time inversion is satisfied for the same values of α in both equations:

$$\alpha = \frac{1}{3}, \frac{2}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{7}, \frac{2}{7}, \ldots, \frac{6}{7}, \frac{1}{9}, \ldots,$$

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*The fractional Dirac equation is not invariant under time traslations due to the nonlocal behaviour of the fractional time derivative.*

# OTHER FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTERNAL DEGREES OF FREEDOM:

## *The 1/3-root of the Wave and Diffusion Equations*

- **Wave Equation:**

  $$P\,\partial_t^{\,2/3}\varphi \;+\; Q\,\partial_x^{\,2/3}\varphi = 0$$

- **Diffusion Equation:**

  $$P\,\partial_t^{\,1/3}\varphi \;+\; Q\,\partial_x^{\,2/3}\varphi = 0$$

- $P^3 = I$ , $Q^3 = -I$
- $PPQ + PQP + QPP = 0$ ; $QQP + QPQ + PQQ = 0$,

- **A possible realization is in terms of the matrices 3x3 associated to the *Silvester Algebra:***

**Where:**

$$P=\begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} \qquad Q=\Omega\begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}$$

**being $\omega$ a cubic root of unity and $\Omega$ a cubic root of the negative unity.**

- **$\varphi$ has three components**

Muchas Gracias
por Vuestro Interés