On the finite time extinction phenomenon for some nonlinear fractional evolution equations

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Outline of Topics

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- A fractional evolution boundary value problem
- Stabilization of a solution
- Extinction in finite time

Our Problem
- Aim of the work

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- Numerical examples

The finite time extinction phenomenon
- The main Theorem
- Sketch of the proof
- Remarks

Conclusions

A fractional evolution boundary value problem

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a general open set.
Let $Q_\infty = \Omega \times (0, +\infty)$, $\Sigma_\infty = \partial\Omega \times (0, +\infty)$, and consider a fractional evolution boundary value problem formulated as follows:

$$\left\{ \begin{array}{rcll} a_1 \frac{\partial u}{\partial t} + a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} + Au & = & f(x,t) & \text{in } Q_\infty, \\ Bu & = & g(x,t) & \text{on } \Sigma_\infty, \\ u(x,0) & = & u_0(x) & \text{in } \Omega. \end{array} \right. \quad (1)$$

- $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$
- $\partial^\alpha / \partial t^\alpha$ is the Riemann-Liouville fractional derivative:

$$\frac{\partial}{\partial t^\alpha} u(x,t) = \frac{\partial}{\partial t} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha} d\tau$$

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- Au denotes a nonlinear operator (usually in terms of u and the partial differentials of u),
- Bu denotes a boundary operator
- the data $f(x,t), g(x,t)$ and $u_0(x)$ are given functions
- for simplicity, A and B are assumed to be *autonomous operators*, i.e., with time independent coefficients.

Stabilization of a solution: main task

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- it is usually assumed that:

$$f(x,t) \to f_\infty(x) \quad \text{and} \quad g(x,t) \to g_\infty(x) \quad \text{as } t \to +\infty,$$

in some functional spaces

Stabilization of a solution: main task

In the study of the stabilization of solutions:

- the main task is to prove that

$$u(x,t) \to u_\infty(x) \text{ as } t \to +\infty,$$

  in some topology of a suitable functional space, with $u_\infty(x)$ solution of

$$\begin{cases} Au_\infty = f_\infty(x) & \text{in } \Omega, \\ Bu_\infty = g_\infty(x) & \text{on } \partial\Omega. \end{cases}$$

- This has been the most recurrent approach in the literature:

♠ Ph. Clément, R.C. MacCamy, J.A. Nohel, *Asymptotic Properties of Solutions of Nonlinear Abstract Volterra Equations*, J. Int. Eq., 3, 185-216, 1981.

A stronger property: extinction in finite time

- Starting by assuming that $A0 = 0$, $B0 = 0$ and

$$\begin{aligned} f(x,t) = 0 \quad \forall\, t \geq T_f, \\ g(x,t) = 0 \quad \forall\, t \geq T_g, \end{aligned}$$

for some $T_f < \infty$ and $T_g < \infty$

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  for some $T_f < \infty$ and $T_g < \infty$

- we want to arrive to the following natural phenomenon of the *extinction in finite time:*

Definition

Let u be a solution of the evolution boundary value problem (1). We will say that $u(x,t)$ possesses the property of extinction in a finite time if there exists $t^* < \infty$ such that

$$u(x,t) \equiv 0 \text{ on } \Omega, \ \forall\, t \geq t^*.$$

The problem under study

Let us consider the following family of general problems:

$$(\mathcal{P}) \begin{cases} a_1 \frac{\partial u}{\partial t} + a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} - \mathrm{div}\left(|\nabla u|^{p-2} \nabla u\right) + \beta(u) = f & \text{in} \quad Q_\infty \\ u = 0 & \text{on} \ \Sigma_\infty \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

where $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$ and $p > 1$.

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where $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$ and $p > 1$.

- In actual fact, if $a_1 = 0$ the initial condition must be understood as follows:

$$\lim_{t\to 0}\Gamma(\alpha)t^{1-\alpha}u(x,t)=\lim_{t\to 0}(I_t^{1-\alpha}u)(x,t)=u_0(x).$$

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where $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$ and $p > 1$.

- When $p = 2$, $\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = \Delta u$
- $\beta(u)$ is the equivalent of the "feedback" term in the control theory.

The problem under study

Let us consider the following family of general problems:

$$(\mathcal{P}) \begin{cases} a_1 \frac{\partial u}{\partial t} + a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} - \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + \beta(u) = f & \text{in} \quad Q_\infty \\ u = 0 & \text{on } \Sigma_\infty \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases}$$

where $a_1 \geq 0$, $a_\alpha > 0$, $\alpha \in (0,1)$ and $p > 1$.

- These problems arise in many contexts, as e.g. the study of the nonlinear reaction-diffusion equation with absorption, or in the heat conduction in materials with memory. See e.g.:
  - J.W. Nunziato, *On heat conduction in materials with memory*, Q. Appl. Math., 29, 187-304, 1971.

Aims

Under suitable conditions, we shall prove that the solution to $(\mathcal{P})$ satisfies an integral energy inequality leading to its extinction in a finite time.

Aims

Concretely:

- We will first prove the occurrence of the extinction in finite time for the problem (1) with $a_1 > 0$ and $a_\alpha > 0$.

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- We will first prove the occurrence of the extinction in finite time for the problem (1) with $a_1 > 0$ and $a_\alpha > 0$.

- Then, we will pass to consider the limit problem obtained when $a_1 = 0$ and $a_\alpha > 0$.

  This is the most extraordinary case, since we prove that the finite time extinction phenomenon still appears, even with a non-smooth profile near the extinction time.

Lemma1

Lemma

Let $\alpha \in (0,1)$ *and* $u \in C^0([0,T]:\mathbb{R})$, $u' \in L^1(0,T:\mathbb{R})$ *and* u *monotone. Then*

$$2\,u(t)\frac{d^\alpha u}{dt^\alpha}(t) \geq \frac{d^\alpha u^2}{dt^\alpha}(t), \qquad \text{a.e.}\ \ t \in (0,T]. \tag{2}$$

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Remarks:

- the inequality (2) can be trivially checked if $\alpha = 1$.
- we conjecture that inequality (2) still holds true under weaker hypothesis on u (avoiding the monotonicity).

- Inequality (2) allows to conclude the monotonicity (or accretiveness) of the fractional differential operator in a very direct way.

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- The proof of the monotonicity has been already provided in the literature by means of very sophisticated arguments. See e.g:

  - G. Gripenber, *Volterra integro-differential equations with accrettive nonlinearity*, J. Differential Eqs., 60, 57-79, 1985
  - PH. Clément and J. Prüss, *Completely positive measures and Feller semigroups*, Math. Ann.,287, 73-105, 1990
  - PH. Clément and S.O. Londen, *On the sum of fractional derivatives and m-accretive operators*, in Partial Differential Equation Models in Physics and Biology, Vol.82, G. Lumer and S. Nicaise and B.W. Schulze (Eds.), Akademie Verlag, 91-100, 1994

Lemma2: a more general version of Lemma 1

Lemma

Given the Hilbert space H, let $\alpha \in (0,1)$ and $u \in L^{\infty}(0,T:H)$ such that $\frac{d^{\alpha}}{dt^{\alpha}}u \in L^{1}(0,T:H)$. Assume that $\|u(\cdot)\|_{H}$ is non-increasing (i.e. $\|u(t_2)\|_{H} \leq \|u(t_1)\|_{H}$ for a.e. $t_1, t_2 \in (0,T)$ such that $t_1 \leq t_2$). Then, there exists $k(\alpha) > 0$ such that for almost every $t \in (0,T)$ we have that

$$\left(u(t), \frac{d^{\alpha}}{dt^{\alpha}}u(t)\right)_{H} \geq k(\alpha)\, \frac{d^{\alpha}}{dt^{\alpha}}\|u(t)\|_{H}^{2}. \tag{3}$$

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$$\left(u(t), \frac{d^\alpha}{dt^\alpha}u(t)\right)_H \geq k(\alpha)\ \frac{d^\alpha}{dt^\alpha}\|u(t)\|_H^2. \tag{3}$$

Lemma2: a more general version of Lemma 1

$$\left(u(t), \frac{d^{\alpha}}{dt^{\alpha}} u(t)\right)_H \geq k(\alpha)\ \frac{d^{\alpha}}{dt^{\alpha}} \|u(t)\|_H^2 .$$

Lemma2: a more general version of Lemma 1

$$\left(u(t), \frac{d^\alpha}{dt^\alpha} u(t) \right)_H \geq k(\alpha)\ \frac{d^\alpha}{dt^\alpha} \|u(t)\|_H^2 .$$

- Inequality (4) directly implies $\frac{d^\alpha}{dt^\alpha} \|u(t)\|_H^2 \in L^1(0, +\infty)$, which is not straightforward to see
- In the literature, it had been already shown (by many authors) that:

$$\int_0^t u(t) \frac{d^\alpha}{dt^\alpha} u(t) dt \geq \int_0^t |u(t)|^2 dt$$

♠ Ph. Clément, R.C. MacCamy, J.A. Nohel, *Asymptotic Properties of Solutions of Nonlinear Abstract Volterra Equations*, J. Int. Eq., 3, 185-216, 1981.

Numerical examples

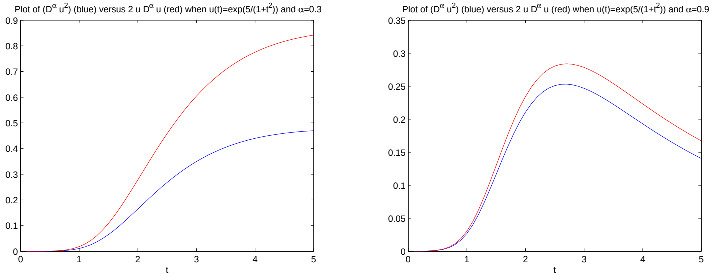

Figure: Function $\frac{d^\alpha u^2}{dt^\alpha}(t)$ vs. $2u(t)\frac{d^\alpha u}{dt^\alpha}(t)$ for $t \in (0,5].$, $u(t) = e^{5/(1+t^2)}$ and different values of α.

Numerical examples

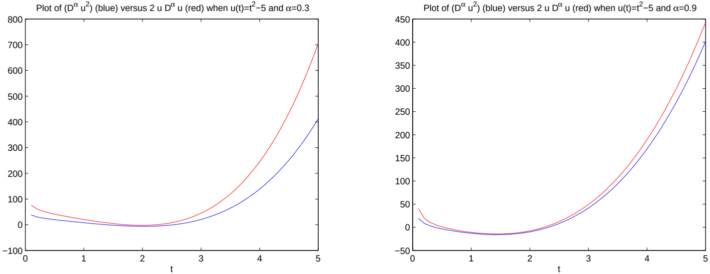

Figure: Function $\frac{d^\alpha u^2}{dt^\alpha}(t)$ vs. $2u(t)\frac{d^\alpha u}{dt^\alpha}(t)$ for $t \in (0,5]$., $u(t) = t^2 - 5$ and different values of α.

Numerical examples

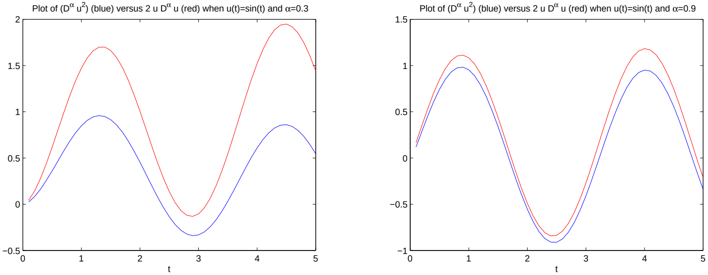

Figure: Function $\frac{d^{\alpha} u^2}{dt^{\alpha}}(t)$ vs. $2u(t)\frac{d^{\alpha} u}{dt^{\alpha}}(t)$ for $t \in (0, 5].$, $u(t) = \sin(t)$ and different values of α.

The finite time extinction phenomenon

Theorem

Let $\beta(\cdot)$ be any nondecreasing continuous function such that $\beta(0) = 0$.
Then, for any $f \in L^1_{\mathrm{loc}}(0, +\infty : L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, there exists a weak solution of the problem $(\mathcal{P})$.
Assume also that $\beta(s) = |s|^{\sigma-1}s$ for some $\sigma > 0$ such that either $p < 2$ and $\sigma > 0$ arbitrary, or $\sigma < 1$ and $p > 1$ arbitrary.
Additionally, let $u_0 \in H^2(\Omega)$, $u_0 \in L^{2\sigma}(\Omega)$ and $f \in H^1_{\mathrm{loc}}(0, +\infty : L^2(\Omega))$ satisfying that $\exists t_f \geq 0$ such that $f(x, t) \equiv 0$ a.e. $x \in \Omega$ and a.e. $t > t_f$.
Then, there exists $t_0 \geq t_f \geq 0$ such that $u(x, t) \equiv 0$ for a.e. $x \in \Omega$ and for any $t \geq t_0$.

Proof of the existence

- The existence of a weak solution $u \in C([0, +\infty) : L^2(\Omega))$ can be deduced from the abstract results on Volterra intregro-differential equations with accretive operators.

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♠ A. Friedman, *On integral equations of the Volterra type, J. Analyse Math.*, 11, 381-413, 1963.

♠ $\vdots$

♠ S. Bonaccorsi and M. Fantozzi, *Volterra integro-differential equations with accretive operators and non-autonomous perturbations*, Journal of Integral Equations and Applications, 18, 437-470, 2006

Proof of the existence

- The existence of a weak solution $u \in C([0,+\infty) : L^2(\Omega))$ can be deduced from the abstract results on Volterra intregro-differential equations with accretive operators.

- The operator $G(u(t)) = -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + \beta(u)$ is m-accretive (or, equivalently, maximal monotone) in $H = L^2(\Omega)$, as it is already well known in the literature.

♠ J.I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Elliptic equations*, Research Notes in Mathematics N.106, Vol.1, Pitman, London, 1985.

An Energy Method. Case $a_1 > 0$

- We define the energy function

$$y(t) := \int_{\Omega} u(x,t)^2 dx = \|u(\cdot,t)\|^2_{L^2(\Omega)}, \tag{4}$$

An Energy Method. Case $a_1 > 0$

- We define the energy function

$$y(t) := \int_\Omega u(x,t)^2 dx = \|u(\cdot,t)\|^2_{L^2(\Omega)}, \qquad (4)$$

- multiplying by u and integrating on Ω the equation appearing in $(\mathcal{P})$, we get, due to the Sobolev, Hölder and Young inequalities:

$$\frac{a_1}{2}\frac{dy}{dt} + a_\alpha \int_\Omega \frac{\partial^\alpha u}{\partial t^\alpha}(x,t)\, u(x,t) dx + C\, y(t)^\nu \leq 0, \qquad (5)$$

for some $C > 0$ and $\nu \in (0,1)$ (this is implied by the hypothesis on σ and p) and for a.e. $t \in (t_f, +\infty)$.

An Energy Method. Case $a_1 > 0$

- Also, we know that [Clement and Prüss,1990] the operator:

$$u \mapsto a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} \tag{6}$$

generates a contraction semigroup in $L^2(\Omega)$.

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- So, since $a_1 > 0$ we get that, for any $t \geq t_f$, the application

$$t \mapsto y(t)$$

is non increasing, $y \in C([t_f, +\infty])$ and $\frac{d^\alpha y}{dt^\alpha} \in L^1(t_f, T)$ (indeed, the weak solution is a strong solution too, thanks to the regularity required on u_0 and f).

An Energy Method. Case $a_1 > 0$

- Also, we know that [Clement and Prüss,1990] the operator:

$$u \longmapsto a_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} \tag{6}$$

generates a contraction semigroup in $L^2(\Omega)$.

- Therefore, we are in conditions as to apply Lemma 2, and we get:

$$\begin{cases} \frac{a_1}{2}\frac{dy}{dt} + \frac{a_\alpha}{2}\frac{d^\alpha y}{dt^\alpha}(t) + C\, y(t)^\nu \leq 0 \quad \text{on } (t_f, +\infty) \\ y(t_f) = Y_0. \end{cases} \tag{7}$$

An Energy Method. Case $a_1 > 0$

- Moreover, since the semigroup generated by the operator (6) is positive [although it is non local] (Clement and Prüss, 1990), we have that:

$$0 \leq y(t) \leq Y(t) \qquad \text{for any } t \in [t_f, +\infty), \tag{8}$$

where $Y(t)$ is a *supersolution*, i.e, $Y(t)$ satisfies the inequality:

$$\begin{cases} \frac{a_1}{2}\frac{dY}{dt} + \frac{a_\alpha}{2}\frac{d^\alpha Y}{dt^\alpha}(t) + C\,Y(t)^\nu \geq 0 \quad \text{on } (t_f, +\infty) \\ Y(t_f) \geq Y_0. \end{cases} \tag{9}$$

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- Our conclusion comes from the fact that we can construct $Y(t)$ satisfying (9) and such that $Y(t) \equiv 0\ \forall\ t \geq t_Y$, for some $t_Y > t_f$.

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- For instance,

$$Y(t) = k\,(t_Y - t)_+^{\frac{1}{1-\nu}}$$

for some $t_Y > t_f$ and some $k > 0$.

An Energy Method. Case $a_1 = 0$

- Let u_ε be the solution of $(\mathcal{P}_\varepsilon)$ when $a_1 = \varepsilon$, $\varepsilon > 0$.
- We can prove that:

$$u_\varepsilon \longrightarrow u^* \quad \text{in } L^2(0, +\infty : L^2(\Omega)) \quad \text{when } \varepsilon \longrightarrow 0,$$

- so the application

$$t \mapsto y^*(t) := \|u^*(\cdot, t)\|^2_{L^2(\Omega)}$$

is also decreasing.

An Energy Method. Case $a_1 = 0$

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- so the application

$$t \mapsto y^*(t) := \|u^*(\cdot, t)\|^2_{L^2(\Omega)}$$

is also decreasing.

- Then, we can apply Lemma 2 and write for y^*:

$$\begin{cases} \frac{a_\alpha}{2} \frac{d^\alpha y^*}{dt^\alpha}(t) + C\, y^*(t)^\nu \leq 0 \quad \text{on } (t_f, +\infty) \\ y(t_f) = W_0. \end{cases} \tag{10}$$

An Energy Method. Case $a_1 = 0$

- The conclusion, as before, comes now from the fact that we can construct a supersolution $W(t)$ satisfying:

$$\begin{cases} \frac{a_\alpha}{2}\frac{d^\alpha W}{dt^\alpha}(t) + C\,W(t)^\nu \geq 0 \quad \text{on } (t_f, +\infty) \\ W(t_f) = W_0, \end{cases} \tag{11}$$

and such that $W(t) \equiv 0 \;\forall\, t \geq t_W$, for some $t_W > t_F$.

- Indeed, let i.e. $W(t) = h\,(t_W - t)^{\frac{\alpha}{1-\nu}}$ for some $t_W > t_f$ and some $h > 0$.

Remarks

- The decreasing behavior of the norm:

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \leq k\,(t_Y - t)_+^{\frac{1}{1-\nu}} \qquad \forall\ t \geq t_f\ ,\ \ \nu \in (0,1) \quad (12)$$

  when $a_1 > 0$ is actually the same as when the fractional derivative is not included in the problem $(\mathcal{P})$.
- It has to be highlighted that when $\nu > 1$ it is well-known that the solution to problem $(\mathcal{P})$ shows an exponential decay at infinity. However our method allows to estimate trough (12)the rate of this decay, which is impossible to achieve with the stabilization methods.

Remarks

- What is more extraordinary is the decreasing behavior of the norm:

$$\|u^*(\cdot,t)\|^2_{L^2(\Omega)} \leq h\,(t_W - t)_+^{\frac{\alpha}{1-\nu}} \qquad \forall\; t \geq t_f\,.$$

when $a_1 = 0$ as we are dealing with a function $W(t)$ such that $\frac{d^\alpha W}{dt^\alpha}(t) \in L^\infty(0,+\infty)$ whereas $W'(t) \notin L^\infty(0,+\infty)$ although $W'(t) \in L^1(0,+\infty)$.

Conclusions

- This work extends the application of the very fine techniques of nonlinear operators on Banach spaces to the case of nonlinear fractional partial differential equations.

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- This work extends the application of the very fine techniques of nonlinear operators on Banach spaces to the case of nonlinear fractional partial differential equations.

- The **finite time extinction phenomenon** for certain evolution boundary values problems are still valid when the evolution in time is given by an *ordinary derivative jointly with a real order differential operator* (which is compatible with unbounded, but integrable, first time derivatives).

Thank you for your kind attention!