On Bessel Functions in the framework of the Fractional Calculus

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In this paper, we shall show the efficiency of a new technique that uses Riemann-Liouville fractional operators to reduce said differential equations to solutions for basic differential equations, which implies representing the more frequently used special functions in terms of basic functions. This representation will also allow us, in many cases, to extend the integral representation of said special functions.

Introduction

In this paper, we shall show the efficiency of a new technique that uses Riemann-Liouville fractional operators to reduce said differential equations to solutions for basic differential equations, which implies representing the more frequently used special functions in terms of basic functions. This representation will also allow us, in many cases, to extend the integral representation of said special functions.

In this article, we shall only consider, by way of example, the Bessel differential operator and its relationship with the Riemann-Liouville fractional operators, as well as the solution to the Bessel equation.

Summary

1. Preliminary results
   - Fractional operators and properties
   - Properties of $\delta = xD$ operator
   - Solutions of a simple fractional differential equation
2. Main result: Commutative rule
3. Application-Example
   - Solution of the Bessel equation
   - Integral representation of the Bessel function
   - Obtention of second solutions
4. Work for the future

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Preliminary results

In this section we introduce some fractional operators, along with a set of properties that will be of use as we proceed in our discussion (see, for example, Samko, Kilbas, Marichev [1]; McBride [2]; Kilbas, Srivastava, Trujillo [3]; Podlubny [4]).

[1] Samko S. G., Kilbas A. A. and Marichev O. I.
Fractinal Integrals and Derivatives. Theory and Applications.
Gordon and Breach. Yvedon, 1993.

[2] McBride A. C.
Fractinal Calculus and Integral Transforms of Generalized Functions.
Ed. Pitman. Adv. Publ. Program. London, 1979.

[3] Kilbas A. A., Srivastava H. M. and Trujillo J. J.
Theory and Applications of Fractional Differential Equations.
Elsevier. Amsterdam, 2006.

[4] Podlubny I.
Fractinal Differential Equations.
Academic Press. San Diego-Boston-New York-London-Tokyo-Toronto, 1999.

Definition (Riemann-Liouville fractional operators)

Let $\alpha > 0$, with $n-1 < \alpha < n$ and $n \in \mathbb{N}$, $[a, b] \subset \mathbb{R}$ and let f be a suitable real function, for example, it suffices if $f \in L_1(a, b)$. The following definitions are well known:

$$
\begin{aligned}
(I^{\alpha}_{a+} f)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt \qquad (x > a) \\
(D^{\alpha}_{a+} f)(x) &= D^n (I^{n-\alpha}_{a+} f)(x) \qquad (x > a) \\
(I^{\alpha}_{b-} f)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt \qquad (x < b) \\
(D^{\alpha}_{b-} f)(x) &= D^n (I^{n-\alpha}_{b-} f)(x), \qquad (x < b)
\end{aligned}
$$

where D is the usual differential operator.

Definition (Generalized Riemann-Liouville operators)

Under the same conditions for function f, let g be a real function such that its derivative $g'(x)$ on $[a,b]$ is greater than 0. Then:

$$
\begin{aligned}
(I^{\alpha}_{a+;\,g}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(t)f(t)}{(g(x)-g(t))^{1-\alpha}}\,dt \qquad (x>a)\\
(D^{\alpha}_{a+;\,g}f)(x) &= D^n_g(I^{n-\alpha}_{a+;\,g}f)(x) \qquad (x>a)\\
(I^{\alpha}_{b-;\,g}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b \frac{g'(t)f(t)}{(g(t)-g(x))^{1-\alpha}}\,dt \qquad (x<b)\\
(D^{\alpha}_{b-;\,g}f)(x) &= (-1)^n D^n_g(I^{n-\alpha}_{b-;\,g}f)(x), \qquad (x<b)
\end{aligned}
$$

where $D^n_g = \left(\frac{1}{g'(x)}\,D\right)^n$.

In particular, for the function $g(x) = x^m$, $m \in \mathbb{N}$, and $a = 0$ we obtain the following fractional integral operators:

$$
\begin{aligned}
(I_m^\alpha f)(x) &= (I_{0+;\, x^m}^\alpha f)(x) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^x (x^m - t^m)^{\alpha-1} f(t)\, dt^m \\
&= \frac{x^{m\alpha}}{\Gamma(\alpha)} \int_0^1 (1 - z^m)^{\alpha-1} f(xz)\, dz^m \qquad (x > 0) \\
(D_m^\alpha f)(x) &= (D_{0+;\, x^m}^\alpha f)(x) = (-D_m^1)^n (I_m^{n-\alpha} f)(x) \qquad (x > 0).
\end{aligned}
$$

The following two Properties are well known:

Property (Exponents' law)

Let f be a suitable function, for instance, locally integrable or continuous, and $\alpha, \beta > 0$*. Then the following relations hold:*

$$\begin{aligned} (I_{a+}^{\alpha} I_{a+}^{\beta} f)(x) &= (I_{a+}^{\alpha+\beta} f)(x) \\ (I_m^{\alpha} I_m^{\beta} f)(x) &= (I_m^{\alpha+\beta} f)(x). \end{aligned}$$

Property

Let $\beta > -1$ *and* $\alpha > 0$ *(*$n - 1 < \alpha < n$*,* $n \in \mathbb{N}$*). Then*

$$D_m^{\alpha} x^{m\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{m(\beta-\alpha)}$$

Property

Let f be a differentiable real function of order 1 in a certain interval $I \subset \mathbb{R}$, $\gamma \in \mathbb{R}$, and $m \in \mathbb{N}$. Then:

$$(\delta\, x^{\gamma})f(x) = (x^{\gamma}(\gamma + \delta))f(x)$$

Property

Let f be a differentiable real function of order 2 in a certain interval $I \subset \mathbb{R}$. Then:

$$D^2 f(x) = (x^{-2}\delta(\delta - 1))f(x)$$

Property

Let $\alpha > 0$, $m \in \mathbb{N}$, and let f be a suitable function in a certain interval $I \subset \mathbb{R}$ (for instance, $f \in C^1(I)$). Then:

$$(I_m^\alpha \delta)f(x) = ((\delta - m\alpha)I_m^\alpha)f(x)$$

$$(D_m^\alpha \delta)f(x) = ((\delta + m\alpha)D_m^\alpha)f(x)$$

and

$$(I_m^\alpha x^{-m}\delta)f(x) = (x^{-m}\delta I_m^\alpha)(f(x) - f(0))$$

$$(D_m^\alpha x^{-m}\delta)f(x) = (x^{-m}\delta D_m^\alpha)(f(x) - f(0))$$

where I_m^α and D_m^α are the Riemann-Liouville fractional operators.

Property

Let $\gamma > 0$ ($n-1 < \alpha < n$, $n \in \mathbb{N}$) and $m \in \mathbb{N}$. Then, $D_m^\alpha \psi(x) = 0$ if, and only if:

$$\psi(x) = \sum_{k=1}^{n} C_k \, x^{m(\alpha-k)}$$

Summary

Main result: Commutative rule

Theorem

Let f be a differential function of order 2 in a certain interval $I \subset \mathbb{R}$, $x \neq 0$, let L_α be the operator

$$L_\alpha = D^2 + \frac{D}{x} - \frac{\alpha^2}{x^2} = x^{-2}(\delta - \alpha)(\delta + \alpha),$$

and let T^α be the operator

$$T^\alpha = x^\alpha D_2^{\alpha+\frac{1}{2}} \quad \left(\alpha > 0,\ \ n - 1 < \alpha + \frac{1}{2} < n,\ \ n \in \mathbb{N}\right),$$

where $D_2^{\alpha+\frac{1}{2}}$ is the generalized Riemann-Liouville fractional differential operator. Then the following relationship holds:

$$(L_\alpha T^\alpha)(f(x) - f(0)) = (T^\alpha D^2) f(x)$$

Summary

In this section we apply the results from the above Theorem to explicitly obtain the solutions to the Bessel Equation:

$$y'' + \frac{y'}{x} + \left(1 - \frac{\nu^2}{x^2}\right) y = 0 \quad (x > 0).$$

We shall also show that one of the solutions extends the well-known integral representation of the Bessel function for all $\nu \in \mathbb{R}$.

First, we note that the Bessel equation may be written in terms of the aforesaid L_α operator, with $\alpha = |\nu|$, as follows:

$$(L_\alpha + 1)y(x) = 0.$$

Making the change of variable $y(x) = T^\alpha(z(x) - z(0))$, the previous equation yields:

$$(L_\alpha + 1)y(x) = T^\alpha[(D^2 + 1)z(x) - z(0)] = 0,$$

since

$$\begin{aligned} (L_\alpha + 1)y(x) &= ((L_\alpha + 1)T^\alpha)(z(x) - z(0)) \\ &= (L_\alpha T^\alpha + T^\alpha)(z(x) - z(0)) \\ &= T^\alpha[(D^2 + 1)z(x) - z(0)]. \end{aligned}$$

Application-Example
Solution of the Bessel equation

Therefore, we can obtain a solution to the Bessel equation at every point $x = x_0$ (unless $x_0 = 0$, in which case we can obtain a solution in any neighboring of $x_0 = 0$), by simply choosing a solution to the basic differential equation:

$$(D^2 + 1)z(x) = z(0),$$

that is,

$$z(x) = C_1 \sin(x) + z(0) \quad (C_1 \textit{ a real constant}).$$

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We shall choose the solution $z_1(x) = \sin(x)$, which yields the following solutions for the Bessel equation, valid for any real value of ν:

$$\begin{aligned} y_1(x) &= (x^{|\nu|} D_2^{|\nu|+\frac{1}{2}}) \sin(x) \\ &= \frac{2^{1-n}\, x^{|\nu|}}{\Gamma(n-(|\nu|+\frac{1}{2}))} \left(x^{-1}\, D\right)^n \int_0^x (x^2-t^2)^{n-|\nu|-\frac{3}{2}}\, t \;\sin t \; dt, \end{aligned}$$

with $x > 0$ and $n - 1 < |\nu| + \frac{1}{2} < n$.

Aside from a constant, the obtained solution represents the Bessel function of order ν, and, given the appropriate restrictions, matches its integral representation. For example, for $\alpha = |\nu| < \frac{1}{2}$ it holds that:

$$y_1(x) = (x^{\alpha} D_2^{\alpha+\frac{1}{2}} I_2^{\frac{1}{2}} D_2^{\frac{1}{2}})(\sin(x)) = \left(\frac{x^{\alpha}}{2} I_2^{\frac{1}{2}-\alpha} x^{-1} \right)(\cos(x)).$$

Abramowitz M. and Stegun I.A.
Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.
Dover. New York, 1972. Abramowitz M. and Stegun I.A.

Lastly, let us point out that various procedures exist for obtaining a second solution $y_2(x)$ linearly independent from $y_1(x)$ for the Bessel equation.

The most natural method is to reduce the Bessel equation to a first order linear equation, for which we already know a solution, and then to solve that equation directly.

$$\begin{aligned} y_2(x) &= y_1(x) \int \frac{1}{x\,(y_1(x))^2}\,dx \\ &= \left[\left(x^{\alpha} D_2^{\alpha+\frac{1}{2}}\right)\sin x\right] \int \frac{1}{x^{2\alpha+1}\left(D_2^{\alpha+\frac{1}{2}}\sin x\right)^2}\,dx. \end{aligned}$$

Additionally, keeping in mind the aforesaid solutions for a simple fractional differential equation, it directly follows that every solution to

$$(D^2 + 1)z(x) = z(0) + x^{(2\alpha-1)},$$

is also a solution to the Bessel equation, except for $x = 0$, as long as z(x) displays suitable behavior at $x = 0$.

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Finding a particular solution to this new equation is easy given the corresponding restrictions:

$$z_p(x) - z_p(0) = \int_0^x t^{(2\alpha-1)}\,(\sin(t)-\cos(t))\,dt,$$

and then we obtain a second linearly independent solution:

$$y_2(x) = \left(x^{\alpha} D_2^{\alpha+\frac{1}{2}}\right) \int_0^x t^{(2\alpha-1)}\,(\sin(t)-\cos(t))\,dt.$$

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Work for the future

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Our next objective is to apply this technique for other linear or partial differential equations. By example,

- Euler-Poisson-Darboux equation

$$\frac{\delta^2\phi}{\delta r^2} + \frac{2n+1}{r}\frac{\delta\phi}{\delta r} = \frac{\delta^2\phi}{\delta t^2}$$

- Equations with special functions as solutions (Legendre, Laguere, Hermite...)

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THANKS FOR YOUR ATTENTION!!