A FRUITFUL INTERPLAY: FROM NONLOCALITY TO FRACTIONAL CALCULUS

L.Vázquez
Departamento de Matemática Aplicada, Facultad de Informatica,
Universidad Complutense, E-28040, Madrid, Spain;
Centro de Astrobiologia (CSIC -INTA), E-28850, Torrejón de Ardoz, Madrid, Spain
lvazquez@fdi.ucm.es

Abstract The Fractional Calculus represents a natural instrument to model nonlocal phenomena either in space or time. From Physics and Chemistry to Biology, there are many processes that involve different space/time scales. In many problems of the above context the dynamics of the system can be formulated by fractional differential equations which include the nonlocal effects. We give a panoramic view of the problem and show some examples.

Keywords: Nonlocality, Fractional derivative, Diffunors, Diffusion process, Generalized Dirac equation

1. Nonlocal Equations

In the context of the field theory, we have the *local theories* associated to the local couplings where the interaction terms are built up from field quantities referring to the same space-time point. On the other hand, we have the *nonlocal theories* defined by nonlocal couplings where the interaction takes place over a "region" of the space-time characterized by a prescribed function [1].

Up to 1994 a very extensive review about the nonlinear nonlocal wave equations with applications to hydrodynamics, magnetohydrodynamics and plasma can be found in the book of Naumkin and Shishmarev [2].

As an illustration of the nonlocal effects, in [3]-[4] a nonlocal generalization of the standard sine-Gordon equation was studied

$$u_{tt} - u_{xx} = 2\cos\frac{u(x,t)}{2}\int dy f(x-y)\sin\frac{u(y,t)}{2} \qquad (1)$$

F. Kh. Abdullaev and V. V. Konotop (eds.), Nonlinear Waves: Classical and Quantum Aspects, 129–133.

This equation can be associated with DNA models as well as the Frenkel-Kontorova model including long–range interaction. When the kernel defined by $f(x-y)$ is the Dirac Delta we have the classical sine-Gordon equation; thus it is difficult to analyze the nonlocal model using a perturbative approach from the local case. The model (1) has static solutions with zero topological charge that do not exist in the local limit. They could be interpreted as frozen breathers originated by the space averaging of the nonlocality. Other nonlocal wave equations had been studied in [5]-[7].

2. Fractional Calculus

There are different definitions of the fractional derivatives but all of them coincide in the integer case (see e.g. [8]-[10] and [15]). The fractional derivative of a function is not determined by the behavior of the function at a single point, but depends on the values of the function over a entire interval. As an example, we have the following definitions of time and space fractional derivatives:

- *Thetime fractional derivative* of order $\alpha > 0$ for a sufficiently well-behaved *causal* function $u(t)$ is defined as follows

$$\frac{d^\alpha}{dt^\alpha}u(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{1-\alpha-m}\, u^{(m)}(\tau)d\tau \qquad (2)$$

  where $m = 1, 2, ...,$ and $0 \leq m-1 < \alpha \leq m$. This definition requires the absolute integrability of the derivative of order m.

- *The symmetric space fractional derivative* [10] of order $\alpha > 0$ of a sufficiently well-behaved function $u(x), x \in \mathbb{R}$, is defined as the pseudo-differential operator characterized in its Fourier representation by

$$\frac{d^\alpha}{d\mid x\mid^\alpha}u(x) \longrightarrow -\mid\kappa\mid^\alpha \hat{u}(\kappa) \qquad (3)$$

  being $\kappa \in \mathbb{R}$.

3. Framework of Applications

The Fractional Calculus offers a unifying framework for different contexts according to the following basic remarks:

- The freedom in the definition of fractional derivatives allow us to incorporate different types of information.
- The fractional derivatives show algebraic scale properties with non-integer exponents what is relevant in the data analysis.

- A new formulation for the damped systems is possible by using the fractional derivatives [11]-[12].
- The fractional derivatives allow a natural interpolation among differential equations of very different properties as the classical wave and heat equations:

$$\frac{\partial^\alpha}{\partial t^\alpha}\Psi - \frac{\partial^2}{\partial x^2}\Psi = 0 \tag{4}$$

  where $1 \leq \alpha \leq 2$.
- The fractional derivative of a function is given by a definite integral, thus it depends on the values of the function over the entire interval. In this context, the fractional derivatives are suitable for the modelling of systems with long range interactions in space and/or time (memory) and processes with many scales of space ans/or time involved.

The applications range in a wide spectrum of areas [13]-[16]: material sciences(viscoelasticity, polymers,..), circuits, diffusion processes, Biology, Economy, Geology, Astrobiology, traffic problems, data analysis,...etc

4. Internal Degrees of Freedom

It is well known the approach of Dirac to obtain his famous equation from the Klein-Gordon equation [1]. The free Dirac equation can be considered as the square root of the Klein-Gordon equation. In a more general context Morinaga and Nono [17] analyzed the integer s-root of the partial differential equations of the form

$$\sum_{|I|=s} a_I \frac{\partial^{|I|}}{\partial x^I}\phi = \phi \tag{5}$$

The $s-$root is the first order system

$$\sum_{i=1}^{n} \alpha_i \frac{\partial \Phi}{\partial x_i} = \Phi \tag{6}$$

where $\alpha_1, ...\alpha_n$ are matrices. From the physical point of view the α_k describe internal degrees of freedom of the associated system.

In the above context, a natural generalization is to consider the fractional diffusion equations with internal degrees of freedom [18]-[19]. They can be obtained as the s-roots of the standard scalar linear diffusion equation. Thus, we have a possible definition of the square root of the standard diffusion equation (SDE) in one space dimension, $u_t - u_{xx} = 0$, as follows:

$$\left(A\frac{\partial^{1/2}}{\partial t^{1/2}} + B\frac{\partial}{\partial x} \right) \psi(x,t) = 0 \tag{7}$$

where A and B are matrices satisfying the conditions:

$$A^2 = I \quad , \quad B^2 = -I \tag{8}$$

$$\{A,B\} \equiv AB + BA = 0. \tag{9}$$

Here $\psi(x,t)$ is a multicomponent function with at least two scalar space-time components. Also, every scalar component satisfies the SDE. Such solutions can be interpreted as *probability distributions with internal structure associated to internal degrees of freedom of the system.* We could name them **diffunors** in analogy with the spinors in Quantum Mechanics.

We have two possible realizations of the above algebra in terms of real matrices 2×2 associated to the Pauli matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10}$$

and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{11}$$

Other realizations involving complex bi-dimensional matrices are possible, but taking into account the meaning of the reference diffusion equation we only consider the real representations.

The solutions of (7) are related to the SDE in a simple way. As an illustration, let us consider the representation (10), i.e.

$$\psi(x,t) = \begin{pmatrix} \varphi(x,t) \\ \chi(x,t) \end{pmatrix}$$

such that $\chi(x,t) = \pm\varphi(x,t)$. We have two general independent solutions of (7):

$$\varphi(x,t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \varphi(x,t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{12}$$

where $\varphi(x,t)$ is a solution of SDE. The solutions (12) represent two possible probability distributions depending not only on the space and time coordinates, but also on the internal degrees of freedom. This effect could model the diffusion of particles with internal structure.

The equation (7) is not time reversible but it is invariant under space reflection as the underlying SDE. More precisely, in the representation given by (10) a possible representation of the **parity operator** is $P = AP^0$, such that $P^0 : x \longrightarrow -x$.

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