Vol 16 No 3, March 2007
1009-1963/2007/16(03)/0570-08

Chinese Physics

Discrete variational principle and first integrals for Lagrange–Maxwell mechanico-electrical systems*

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(Received 21 September 2005; revised manuscript received 14 September 2006)

This paper presents a discrete variational principle and a method to build first-integrals for finite dimensional Lagrange–Maxwell mechanico-electrical systems with nonconservative forces and a dissipation function. The discrete variational principle and the corresponding Euler–Lagrange equations are derived from a discrete action associated to these systems. The first-integrals are obtained by introducing the infinitesimal transformation with respect to the generalized coordinates and electric quantities of the systems. This work also extends discrete Noether symmetries to mechanico-electrical dynamical systems. A practical example is presented to illustrate the results.

Keywords: discrete, variational principle, first integral, mechanico-electrical systems
PACC: 0220; 0320

1. Introduction

Dynamical systems with symmetries play an important role in the mathematical modelling of a large class of physical and mechanical processes. The symmetries allow one to build the invariants of the systems,[1] that can be useful to integrate the equations of motion, and also to construct suitable numerical integrators for different equations of mathematical physics.[2,3] In this paper, our aim is to provide a technique to find the equivalent of first integrals for a discrete system, as an extension of the continuous case. Arising from different approaches, discrete variational principles and first integrals for discrete mechanics have been considered over many years. The theory of discrete variational mechanics goes back to the 1960s, when Jordan and Polak[4] first treated them in the optimal control literature. Cadzow[5] motivated and discussed discrete calculus of variations and obtained the discrete Euler–Lagrange equations. Logan[6] obtained first integrals, using the discrete calculus of variation and a discrete Noether theorem, and studied the multi-dimensional as well as the higher-order extensions. Maeda[7,8] analysed the canonical structure and the symmetries for discrete systems, and extended Noether's theorem in the discrete case. Lee[9] was the first to consider time also as a discrete dynamical variable. These ideas were extended by Veselov,[10,11] Moser and Veselov[12] for integrable systems. Jaroszkiewiez and Norteo[13−15] applied this to some discrete mechanical models, including systems of particles, classical fields and quantum theory. The variational point of view and the numerical implementation of discrete mechanics have been developed by Wendlandt and Marsden,[16] Kane *et al*,[17] and Bobenko and Suris.[18] Following this, Marsden *et al*,[19] and Bokenko and Suris[20] considered symmetry reductions of discrete Lagrangian mechanics, discrete Lagrangian reduction, etc. Kane *et al*[21] extended variational integration algorithms to discrete, dissipative, mechanical systems. In the work of Marsden and West,[22] a comprehensive and unified view of much of these works both on discrete me-

*Project supported by State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences and the National Natural Science Foundation of China (Grant Nos 10672143 and 10471145) and the Natural Science Foundation of Henan Province Government, China (Grant Nos 0311011400 and 0511022200).
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chanics and on numerical integration methods for mechanical systems can be found. Recently, Guo, Wu *et al*[23−27] have presented many results on difference discrete variational principles, Euler–Lagrange cohomology, symplectic and multisymplectic structures and total variation in Hamiltonian formalism, symplectic-energy integrators, etc. Zhang and Chen *et al*[28−30] have recently obtained many results on discrete variation principles and first integrals of dynamical systems.

The study of discrete variational principles and of first integrals for mechanical systems has been for many years a field with an intense research activity. The increasing interest in the subject is mainly due to its dual character. On the one hand, discrete variational principles and first integrals of mechanical systems allow for the construction of integration schemes that turn out to be numerically competitive in many problems. On the other hand, many of the geometric properties of mechanical systems in the continuous case admit an appropriate counterpart in the discrete setting, which makes it a rich area to be explored. Both aspects play a key role in explaining the good behaviour shown by the integrators in simulating many different systems.

Mechanico-electrical systems are those in which a mechanical process and an electromagnetic one are coupled to each other. Solving a mechanico-electrical systems is in general a difficult task, since we may expect the presence of nonlinear terms. These systems have many applications, and it is relevant, thus, to build tools that can provide numerical solutions. Besides, the discrete variational techniques and the related integration methods appear as an indispensable tool in the domain of modern engineering technology. The purpose of this paper is to extend the discrete variational principle and the method for first integrals used in mechanical systems to mechanico-electrical systems. In the first place, we use a discrete Lagrangian to define a discrete action. This includes defining generalized coordinates and generalized electric quantities. Then, we obtain a discrete variational principle and the corresponding discrete Euler–Lagrange equations for the Lagrange–Maxwell mechanico-electrical systems. And finally, we derive the method of building first integrals by using infinitesimal transformations with respect to the generalized coordinates and the generalized electric quantities. In future works, we will test the algorithms thus obtained in some relevant examples.

2. Lagrange–Maxwell equations for Lagrange–Maxwell mechanico-electrical systems

We present now the standard procedure of deriving the equations of motion for a mechanico-electrical system (see for instance Ref.[31]). In a mechanico-electrical system, mechanical and electromagnetic processes are coupled to each other: let us consider a mechanico-electrical system composed of some particles for the mechanical part, described by their generalized n-dimensional coordinate-vectors $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ and by some mutual-interaction potential $V(\boldsymbol{q})$, and m return electric circuits, that constitute the electrodynamical part, consisting of line conductors and capacitors. We assume that there is no connection among the electric currents of the different return circuits and that the electromagnetic processes of the return circuits are not independent. For circuit k, i_k denotes the current, u_k the electric potential, e_k ($\dot{e}_k = i_k$) the charge in the capacitor, R_k denotes the resistance, and C_k denotes the capacitance. With this, the Lagrangian of the mechanico-electrical system is given by

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{e}, \dot{\boldsymbol{e}}) - V(\boldsymbol{q}) + W_{\rm m}(\boldsymbol{q}, \dot{\boldsymbol{e}}) - W_{\rm e}(\boldsymbol{q}, \dot{\boldsymbol{e}})\,, \quad (1)$$

where

$$W_{\rm e} = \frac{1}{2}\sum_{k=1}^{m}\frac{e_k^2}{C_k}\,, \quad W_{\rm m} = \frac{1}{2}\sum_{k=1}^{m}\sum_{r=1}^{m} L_{kr} i_k i_r\,, \quad (2)$$

are, respectively, the electric and the magnetic field energy of the circuits. In Eq.(2), $C_k = C_k(\boldsymbol{q})$ is the capacitance of the kth circuit, $L_{kr} = L_{kr}(\boldsymbol{q})$ $(k \neq r)$ is the mutual inductance between the kth and the rth circuits, and L_{kk} is the self-inductance of the kth return circuit.

The equations of motion, or Lagrange–Maxwell equations, for the system are

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} + \frac{\partial F}{\partial \dot{e}_k} = u_k\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s} = Q''_s\,,\\
&(k = 1, \ldots, m; s = 1, \ldots, n). \quad (3)
\end{aligned}$$

They conform a set of $n+m$ ordinary differential equations. The force F is given by

$$F = F_{\rm e}(\dot{\boldsymbol{e}}) + F_{\rm m}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (4)$$

where $F_{\rm e}$ is the lead-through electric dissipative function:

$$F_{\rm e}=\frac{1}{2}\sum_{k=1}^{m}R_k i_k^2=\frac{1}{2}\sum_{k=1}^{m}R_k\dot{e}_k^2\,, \tag{5}$$

and $F_{\rm m}$ is the dissipative function of the viscous frictional damping force. Both $-\partial F/\partial\dot{q}_s$ and $-\partial F/\partial\dot{e}_k$ correspond to dissipative forces, Q''_s is the s-component of a non-conservative general force Q'', and u_k is the general electromotive force of the kth circuit.

3. Discrete variational principle and Euler–Lagrange equations for a Lagrange–Maxwell mechanico-electrical system

Let us consider an $(n+m)$-dimensional configuration space Q. The discrete Lagrangian and the discrete dissipative function for a mechanico-electrical system are smooth maps:

$$L_{\rm d}:Q\times Q\rightarrow \boldsymbol{R},\quad F_{\rm d}:Q\times Q\rightarrow\boldsymbol{R}. \tag{6}$$

The time step information is contained in $L_{\rm d}$ which is a function of the four values $(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)$. For any positive integers N and J, the action sum is the map $S_{\rm d}:Q^{N+J+2}\rightarrow\boldsymbol{R}$ defined by

$$S_{\rm d}=\sum_{k=1}^{N}\sum_{l=1}^{J}L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)\,, \tag{7}$$

where $(\boldsymbol{q}_{k-1},\boldsymbol{e}_{l-1})\in Q$ and $(\boldsymbol{q}_k,\boldsymbol{e}_l)\in Q$, for all the considered values of k,l. The action sum is the discrete analogue of the action integral for a continuous mechanico-electrical dynamical system.

Let us consider, as an example, a continuous mechanico-electrical system with a Lagrangian of the standard form

$$L(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{e},\dot{\boldsymbol{e}})=\frac{1}{2}\dot{\boldsymbol{q}}^{\rm T}M\dot{\boldsymbol{q}}-V(\boldsymbol{q})+\frac{1}{2}\boldsymbol{e}^{\rm T}C^{-1}\boldsymbol{e}+\frac{1}{2}\dot{\boldsymbol{e}}^{\rm T}L_1\dot{\boldsymbol{e}}\,, \tag{8}$$

where M and L_1 are symmetric positive-definite mass matrices, C is the diagonal $m\times m$ matrix having C_k as k,k-element, $\boldsymbol{q}=(q_1,\ldots,q_n)\in\boldsymbol{R}^n$, $\boldsymbol{e}=(e_1,\ldots,e_m)\in\boldsymbol{R}^m$, and $(\boldsymbol{q},\boldsymbol{e})\in\boldsymbol{R}^{n+m}=Q$. Different choices for the discrete Lagrangian give rise to the same discrete Euler–Lagrange equations. Using an interpolation parameter $0\le\alpha\le1$, we have chosen a symmetrized discrete Lagrangian, $L_{\rm d}:Q\times Q\rightarrow\boldsymbol{R}$, of the form

$$\begin{aligned}L_{\rm d}^{\alpha}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)=&\frac{h}{2}L\Big((1-\alpha)\boldsymbol{q}_{k-1}+\alpha\boldsymbol{q}_k,\frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h},(1-\alpha)\boldsymbol{e}_{l-1}+\alpha\boldsymbol{e}_l,\frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big)\\&+\frac{h}{2}L\Big(\alpha\boldsymbol{q}_{k-1}+(1-\alpha)\boldsymbol{q}_k,\frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h},\alpha\boldsymbol{e}_{l-1}+(1-\alpha)\boldsymbol{e}_l,\frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big),\end{aligned} \tag{9}$$

where $h\in\boldsymbol{R}^+$ is the time step. We also chose a symmetrized discrete dissipative function, $F_{\rm d}:Q\times Q\rightarrow\boldsymbol{R}$, of the form

$$\begin{aligned}F_{\rm d}^{\alpha}(\boldsymbol{q}_{l-k},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)=&\frac{h}{2}F\Big((1-\alpha)\boldsymbol{q}_{k-1}+\alpha\boldsymbol{q}_k,\frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h},\frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big)\\&+\frac{h}{2}F\Big(\alpha\boldsymbol{q}_{k-1}+(1-\alpha)\boldsymbol{q}_k,\frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h},\frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big).\end{aligned} \tag{10}$$

For instance, substituting Eq.(8) in Eq.(9), and simplifying, we obtain the discrete Lagrangian:

$$\begin{aligned}L_{\rm d}^{\alpha}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)=&\frac{h}{2}\Big[\Big(\frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h}\Big)^{\rm T}M\Big(\frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h}\Big)-V((1-\alpha)\boldsymbol{q}_{k-1}+\alpha\boldsymbol{q}_k)\\&-V\big(\alpha\boldsymbol{q}_{k-1}+(1-\alpha)\boldsymbol{q}_k\big)+\Big(\frac{\boldsymbol{e}_{l-1}-\boldsymbol{e}_l}{h}\Big)^{\rm T}L_1\Big(\frac{\boldsymbol{e}_{l-1}-\boldsymbol{e}_l}{h}\Big)\\&+\frac{1}{2}(\boldsymbol{e}_{l-1}^{\rm T}C^{-1}\boldsymbol{e}_{l-1}+\boldsymbol{e}_l^{\rm T}C^{-1}\boldsymbol{e}_l)+\alpha(\alpha-1)(\boldsymbol{e}_l-\boldsymbol{e}_{l-1})^{\rm T}C^{-1}(\boldsymbol{e}_l-\boldsymbol{e}_{l-1})\Big]\,.\end{aligned} \tag{11}$$

Keeping in mind that the (continuous) integral variational principle for mechanico-electrical systems is given by Ref.[32]

$$\delta\int L\big(\boldsymbol{q}(t),\boldsymbol{e}(t),\dot{\boldsymbol{q}}(t),\dot{\boldsymbol{e}}(t)\big)\,{\rm d}t+\int\Big(\boldsymbol{Q}''-\frac{\partial F}{\partial\dot{\boldsymbol{q}}}\Big)\circ\delta\boldsymbol{q}\,{\rm d}t+\int\Big(\boldsymbol{u}-\frac{\partial F}{\partial\dot{\boldsymbol{e}}}\Big)\circ\delta\boldsymbol{e}\,{\rm d}t=0\,, \tag{12}$$

where $\circ$ stands for the scalar products, we define the discrete variational principle as

$$\begin{aligned}
&\delta \sum L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l) + \sum \Big[\boldsymbol{Q}''^{-}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k) - \frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|^{-}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)\Big] \circ \delta\boldsymbol{q}_{k-1}\\
&+ \sum \Big[\boldsymbol{Q}''^{+}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k) - \frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|^{+}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)\Big] \circ \delta\boldsymbol{q}_{k}\\
&+ \sum \Big[\boldsymbol{u}^{-}_{\rm d}(\boldsymbol{e}_{l-1},\boldsymbol{e}_l) - \frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|^{-}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)\Big] \circ \delta\boldsymbol{e}_{l-1}\\
&+ \sum \Big[\boldsymbol{u}^{+}_{\rm d}(\boldsymbol{e}_{l-1},\boldsymbol{e}_l) - \frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|^{+}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)\Big] \circ \delta\boldsymbol{e}_{l} = 0\,,
\end{aligned} \tag{13}$$

where $L_{\rm d}$ is the discrete Lagrangian, $\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|^{-}_{\rm d}$, $\frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|^{-}_{\rm d}$, $\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|^{+}_{\rm d}$, and $\frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|^{+}_{\rm d}$ are the left and right discrete dissipative forces respectively, given by

$$\begin{aligned}
\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|^{-}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l) = \frac{h}{2}\Big[&\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big((1-\alpha)\boldsymbol{q}_{k-1}+\alpha\boldsymbol{q}_k, \frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h}, \frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big)\\
&+\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big(\alpha\boldsymbol{q}_{k-1}+(1-\alpha)\boldsymbol{q}_k, \frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h}, \frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big)\Big]\,,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|^{+}_{\rm d}(\boldsymbol{q}_{k},\boldsymbol{q}_{k+1},\boldsymbol{e}_{l},\boldsymbol{e}_{l+1}) = \frac{h}{2}\Big[&\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big((1-\alpha)\boldsymbol{q}_{k}+\alpha\boldsymbol{q}_{k+1}, \frac{\boldsymbol{q}_{k+1}-\boldsymbol{q}_{k}}{h}, \frac{\boldsymbol{e}_{l+1}-\boldsymbol{e}_{l}}{h}\Big)\\
&+\frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big(\alpha\boldsymbol{q}_{k}+(1-\alpha)\boldsymbol{q}_{k+1}, \frac{\boldsymbol{q}_{k+1}-\boldsymbol{q}_{k}}{h}, \frac{\boldsymbol{e}_{l+1}-\boldsymbol{e}_{l}}{h}\Big)\Big]\,,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|^{-}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l) = \frac{h}{2}\Big[&\frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big((1-\alpha)\boldsymbol{q}_{k-1}+\alpha\boldsymbol{q}_k, \frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h}, \frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big)\\
&+\frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big(\alpha\boldsymbol{q}_{k-1}+(1-\alpha)\boldsymbol{q}_k, \frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h}, \frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big)\Big]\,,
\end{aligned} \tag{16}$$

$$\begin{aligned}
\frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|^{+}_{\rm d}(\boldsymbol{q}_{k},\boldsymbol{q}_{k+1},\boldsymbol{e}_{l},\boldsymbol{e}_{l+1}) = \frac{h}{2}\Big[&\frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big((1-\alpha)\boldsymbol{q}_{k}+\alpha\boldsymbol{q}_{k+1}, \frac{\boldsymbol{q}_{k+1}-\boldsymbol{q}_{k}}{h}, \frac{\boldsymbol{e}_{l+1}-\boldsymbol{e}_{l}}{h}\Big)\\
&+\frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big(\alpha\boldsymbol{q}_{k}+(1-\alpha)\boldsymbol{q}_{k+1}, \frac{\boldsymbol{q}_{k+1}-\boldsymbol{q}_{k}}{h}, \frac{\boldsymbol{e}_{l+1}-\boldsymbol{e}_{l}}{h}\Big)\Big]\,,
\end{aligned} \tag{17}$$

and $\boldsymbol{Q}''^{-}_{\rm d}$, $\boldsymbol{u}^{-}_{\rm d}$, $\boldsymbol{Q}''^{+}_{\rm d}$ and $\boldsymbol{u}^{+}_{\rm d}$ are the left and right discrete non-conservative forces:

$$\boldsymbol{Q}''^{-}_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k) = \frac{h}{2}\Big[\boldsymbol{Q}''\Big((1-\alpha)\boldsymbol{q}_{k-1}+\alpha\boldsymbol{q}_k, \frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h}\Big) + \boldsymbol{Q}''\Big(\alpha\boldsymbol{q}_{k-1}+(1-\alpha)\boldsymbol{q}_k, \frac{\boldsymbol{q}_k-\boldsymbol{q}_{k-1}}{h}\Big)\Big], \tag{18}$$

$$\boldsymbol{u}^{-}_{\rm d}(\boldsymbol{e}_{l-1},\boldsymbol{e}_l) = \frac{h}{2}\Big[\boldsymbol{u}\Big((1-\alpha)\boldsymbol{e}_{l-1}+\alpha\boldsymbol{e}_l, \frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big) + \boldsymbol{u}\Big(\alpha\boldsymbol{e}_{l-1}+(1-\alpha)\boldsymbol{e}_l, \frac{\boldsymbol{e}_l-\boldsymbol{e}_{l-1}}{h}\Big)\Big], \tag{19}$$

$$\boldsymbol{Q}''^{+}_{\rm d}(\boldsymbol{q}_{k},\boldsymbol{q}_{k+1}) = \frac{h}{2}\Big[\boldsymbol{Q}''\Big((1-\alpha)\boldsymbol{q}_{k}+\alpha\boldsymbol{q}_{k+1}, \frac{\boldsymbol{q}_{k+1}-\boldsymbol{q}_{k}}{h}\Big) + \boldsymbol{Q}''\Big(\alpha\boldsymbol{q}_{k}+(1-\alpha)\boldsymbol{q}_{k+1}, \frac{\boldsymbol{q}_{k+1}-\boldsymbol{q}_{k}}{h}\Big)\Big], \tag{20}$$

$$\boldsymbol{u}^{+}_{\rm d}(\boldsymbol{e}_{l},\boldsymbol{e}_{l+1}) = \frac{h}{2}\Big[\boldsymbol{u}\Big((1-\alpha)\boldsymbol{e}_{l}+\alpha\boldsymbol{e}_{l+1}, \frac{\boldsymbol{e}_{l+1}-\boldsymbol{e}_{l}}{h}\Big) + \boldsymbol{u}\Big(\alpha\boldsymbol{e}_{l}+(1-\alpha)\boldsymbol{e}_{l+1}, \frac{\boldsymbol{e}_{l+1}-\boldsymbol{e}_{l}}{h}\Big)\Big]. \tag{21}$$

The variation of the discrete action is given by

$$\begin{aligned}
\delta \sum L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l) &= \sum_{k=1}^{N}\sum_{l=1}^{J}\Big[\frac{\partial L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)}{\partial \boldsymbol{q}_{k-1}} \circ \delta\boldsymbol{q}_{k-1} + \frac{\partial L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)}{\partial \boldsymbol{q}_{k}} \circ \delta\boldsymbol{q}_{k}\Big]\\
&\quad + \sum_{k=1}^{N}\sum_{l=1}^{J}\Big[\frac{\partial L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)}{\partial \boldsymbol{e}_{l-1}} \circ \delta\boldsymbol{e}_{l-1} + \frac{\partial L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)}{\partial \boldsymbol{e}_{l}} \circ \delta\boldsymbol{e}_{l}\Big]\\
&= \sum_{k=1}^{N-1}\sum_{l=1}^{J-1}\Big[\frac{\partial L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)}{\partial \boldsymbol{q}_{k}} + \frac{\partial L_{\rm d}(\boldsymbol{q}_{k},\boldsymbol{q}_{k+1},\boldsymbol{e}_{l},\boldsymbol{e}_{l+1})}{\partial \boldsymbol{q}_{k}}\Big] \circ \delta\boldsymbol{q}_{k}\\
&\quad + \sum_{k=1}^{N-1}\sum_{l=1}^{J-1}\Big[\frac{\partial L_{\rm d}(\boldsymbol{q}_{k-1},\boldsymbol{q}_k,\boldsymbol{e}_{l-1},\boldsymbol{e}_l)}{\partial \boldsymbol{e}_{l}} + \frac{\partial L_{\rm d}(\boldsymbol{q}_{k},\boldsymbol{q}_{k+1},\boldsymbol{e}_{l},\boldsymbol{e}_{l+1})}{\partial \boldsymbol{e}_{l}}\Big] \circ \delta\boldsymbol{e}_{l}\,,
\end{aligned} \tag{22}$$

since we have to consider the extremal points constant and, thus, $\delta\boldsymbol{q}_0 = \delta\boldsymbol{q}_N = \vec{0}$ and $\delta\boldsymbol{e}_0 = \delta\boldsymbol{e}_J = \vec{0}$. Substituting all this into Eq.(13), and changing the summation index when necessary, as in Eq.(22), we obtain the discrete Euler–Lagrange equations, setting to zero the coefficients of $\delta\boldsymbol{q}_k$ and of $\delta\boldsymbol{e}_l$ for $k = 1, \ldots, N-1$; $l = 1, \ldots, J-1$:

$$\begin{aligned}
&\frac{\partial L_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l)}{\partial \boldsymbol{q}_k} + \frac{\partial L_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1})}{\partial \boldsymbol{q}_k} + \boldsymbol{Q}''^{-}_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) + \boldsymbol{Q}''^{+}_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \\
&\quad - \frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|_\mathrm{d}^{-}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) - \frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|_\mathrm{d}^{+}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) = 0\,, \\
&\frac{\partial L_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l)}{\partial \boldsymbol{e}_l} + \frac{\partial L_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1})}{\partial \boldsymbol{e}_l} + \boldsymbol{u}^{-}_\mathrm{d}(\boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + \boldsymbol{u}^{+}_\mathrm{d}(\boldsymbol{e}_l, \boldsymbol{e}_{l+1}) \\
&\quad - \frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|_\mathrm{d}^{-}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) - \frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|_\mathrm{d}^{+}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) = 0\,.
\end{aligned} \tag{23}$$

It will be useful to have the equations written in the equivalent form:

$$\begin{aligned}
&D_2 L_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + D_1 L_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) + \boldsymbol{Q}''^{-}_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) + \boldsymbol{Q}''^{+}_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \\
&\quad - \frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|_\mathrm{d}^{-}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) - \frac{\partial F}{\partial \dot{\boldsymbol{q}}}\Big|_\mathrm{d}^{+}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) = 0\,, \\
&D_4 L_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + D_3 L_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) + \boldsymbol{u}^{-}_\mathrm{d}(\boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + \boldsymbol{u}^{+}_\mathrm{d}(\boldsymbol{e}_l, \boldsymbol{e}_{l+1}) \\
&\quad - \frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|_\mathrm{d}^{-}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) - \frac{\partial F}{\partial \dot{\boldsymbol{e}}}\Big|_\mathrm{d}^{+}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) = 0\,,
\end{aligned} \tag{24}$$

where D_j represent the gradient with respect to the jth (vectorial) variable:

$$\begin{aligned}
D_2 L_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) &= \frac{\partial L_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l)}{\partial \boldsymbol{q}_k}, \\
D_1 L_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) &= \frac{\partial L_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1})}{\partial \boldsymbol{q}_k}, \\
D_4 L_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) &= \frac{\partial L_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l)}{\partial \boldsymbol{e}_l}, \\
D_3 L_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) &= \frac{\partial L_\mathrm{d}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1})}{\partial \boldsymbol{e}_l}.
\end{aligned} \tag{25}$$

4. First integrals for discrete Lagrange–Maxwell mechanico-electrical systems

For continuous systems, Noether's theory states that a symmetry of the Lagrangian leads to a conserved quantity, also called a first integral. We will now present a discrete version of Noether's theorem and derive a method to build the corresponding discrete first integral. We use the invariance of the discrete Lagrangian for discrete Lagrange–Maxwell mechanico-electrical systems.

Let us consider the following infinitesimal transformations for the discrete coordinates and electric quantities:

$$\boldsymbol{q}^* = \boldsymbol{q} + \varepsilon\boldsymbol{\xi}(\boldsymbol{q}, \boldsymbol{e}), \qquad \boldsymbol{e}^* = \boldsymbol{e} + \varepsilon\boldsymbol{\eta}(\boldsymbol{q}, \boldsymbol{e}), \tag{26}$$

where ε is a small parameter, and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_\mathrm{m})$ are the infinitesimal generators. We first give a definition as follows:

Definition The discrete Lagrangian L_d, for a system with nonconservative force Q_d and general electromotive force $\boldsymbol{u}_\mathrm{d}$, is generalized difference-invariant under the transformation (26) if there exists a function $v(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l)$, defined for each value of $k = 1, \ldots, N-1$ and $l = 1, \ldots, J-1$, such that

$$\delta L_\mathrm{d}[\varepsilon\boldsymbol{\xi}_k(\boldsymbol{q}, \boldsymbol{e}), \varepsilon\boldsymbol{\eta}_l(\boldsymbol{q}, \boldsymbol{e})] = \varepsilon\Big[\frac{\partial F}{\partial \dot{q}}\Big|_\mathrm{d}^{-}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + \frac{\partial F}{\partial \dot{q}}\Big|_\mathrm{d}^{+}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) - \boldsymbol{Q}''^{-}_\mathrm{d}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k)$$

$$- \boldsymbol{Q}_{\mathrm{d}}^{\prime\prime +}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})\Big] \circ \boldsymbol{\xi}(\boldsymbol{q}_k, \boldsymbol{e}_l) + \varepsilon\Big[\frac{\partial F}{\partial \dot{e}}\Big|_{\mathrm{d}}^{-}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + \frac{\partial F}{\partial \dot{e}}\Big|_{\mathrm{d}}^{+}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1})$$
$$- \boldsymbol{u}_{\mathrm{d}}^{-}(\boldsymbol{e}_{l-1}, \boldsymbol{e}_l) - \boldsymbol{u}_{\mathrm{d}}^{+}(\boldsymbol{e}_l, \boldsymbol{e}_{l+1})\Big] \circ \boldsymbol{\eta}(\boldsymbol{q}_k, \boldsymbol{e}_l) + \varepsilon \Delta v(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l)\,, \tag{27}$$

where Δ is the (forward) difference operator, i.e., $\Delta \boldsymbol{q}_k = \boldsymbol{q}_{k+1} - \boldsymbol{q}_k$, $\Delta \boldsymbol{e}_l = \boldsymbol{e}_{l+1} - \boldsymbol{e}_l$, and in general:

$$\delta L_{\mathrm{d}} = D_2 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \delta \boldsymbol{q}_k + D_1 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \delta \boldsymbol{q}_{k-1}$$
$$+ D_4 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \delta \boldsymbol{e}_l + D_3 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \delta \boldsymbol{e}_{l-1}\,. \tag{28}$$

Based on the discrete Lagrange–D'Alembert principle, we can present the following proposition:

Proposition If the discrete Lagrangian is generalized difference invariant under the infinitesimal transformation (26) and the discrete Euler–Lagrange equations (24) hold, then the discrete Lagrange–Maxwell mechanico-electrical system possesses a conserved quantity, or a first integral given by

$$D_1 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \boldsymbol{\xi}(\boldsymbol{q}_k, \boldsymbol{e}_l) + D_3 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \boldsymbol{\eta}(\boldsymbol{q}_k, \boldsymbol{e}_l) + v(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) = \text{const.} \tag{29}$$

Proof From Eq.(27), since in this case $\delta \boldsymbol{q} = \varepsilon \boldsymbol{\xi}$ and $\delta \boldsymbol{e} = \varepsilon \boldsymbol{\eta}$, we have

$$\varepsilon D_2 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \boldsymbol{\xi}(\boldsymbol{q}_k, \boldsymbol{e}_l) + \varepsilon D_1 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \boldsymbol{\xi}(\boldsymbol{q}_{k-1}, \boldsymbol{e}_{l-1})$$
$$+ \varepsilon D_4 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \boldsymbol{\eta}(\boldsymbol{q}_k, \boldsymbol{e}_l) + \varepsilon D_3 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \boldsymbol{\eta}(\boldsymbol{q}_{k-1}, \boldsymbol{e}_{l-1})$$
$$= \varepsilon\Big[\frac{\partial F}{\partial \dot{q}}\Big|_{\mathrm{d}}^{-}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + \frac{\partial F}{\partial \dot{q}}\Big|_{\mathrm{d}}^{+}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) - \boldsymbol{Q}_{\mathrm{d}}^{\prime\prime -}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) - \boldsymbol{Q}_{\mathrm{d}}^{\prime\prime +}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})\Big] \circ \boldsymbol{\xi}(\boldsymbol{q}_k, \boldsymbol{e}_l)$$
$$+ \varepsilon\Big[\frac{\partial F}{\partial \dot{e}}\Big|_{\mathrm{d}}^{-}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + \frac{\partial F}{\partial \dot{e}}\Big|_{\mathrm{d}}^{+}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1}) - \boldsymbol{u}_{\mathrm{d}}^{-}(\boldsymbol{e}_{l-1}, \boldsymbol{e}_l) - \boldsymbol{u}_{\mathrm{d}}^{+}(\boldsymbol{e}_l, \boldsymbol{e}_{l+1})\Big] \circ \boldsymbol{\eta}(\boldsymbol{q}_k, \boldsymbol{e}_l)$$
$$+ \varepsilon \Delta v(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l)\,. \tag{30}$$

Using the fact that Eq.(28) can also be expressed as

$$\delta L_{\mathrm{d}} = \big[D_2 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + D_1 L_{\mathrm{d}}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1})\big] \circ \delta \boldsymbol{q}_k + \Delta\big(- D_1 L_{\mathrm{d}}(q_{k-1}, q_k, e_{l-1}, e_l) \circ \delta \boldsymbol{q}_{k-1}\big)$$
$$+ \big[D_4 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) + D_3 L_{\mathrm{d}}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, \boldsymbol{e}_l, \boldsymbol{e}_{l+1})\big] \circ \delta \boldsymbol{e}_l + \Delta\big(- D_3 L_{\mathrm{d}}(q_{k-1}, q_k, e_{l-1}, e_l) \circ \delta \boldsymbol{e}_{l-1}\big), \tag{31}$$

and substituting Eq.(24) into Eq.(30), and simplifying the result, we finally have

$$\Delta\big[D_1 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \boldsymbol{\xi}(\boldsymbol{q}_k, \boldsymbol{e}_l) + D_3 L_{\mathrm{d}}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l) \circ \boldsymbol{\eta}(\boldsymbol{q}_k, \boldsymbol{e}_l) + v(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k, \boldsymbol{e}_{l-1}, \boldsymbol{e}_l)\big] = 0\,, \tag{32}$$

which implies the result.

5. Numerical example

Figure 1 represents a circuit of an electromotion sensor to record mechanical vibrations. The circuit is composed of a coil, a battery and a resistance. We represent by m the mass of the armature, a denotes the total rigidity coefficient, $L_1 = L_1(x)$ represents the self-induction in the coil, x represents the vertical displacement from the position of the winding L_1, we denote by E the electromotive force of the battery and R is the value of the resistance.

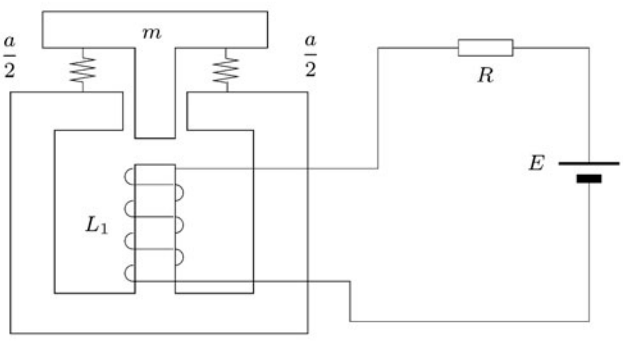

Fig.1.

The mechanical part of the system is described by the displacement of the armature x, while the electrical part is described by the electric quantity q. Considering x and q as our generalized coordinates, the kinetic plus magnetic energy of the system is

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}L_1(x)\dot{q}^2, \tag{33}$$

the potential energy is:

$$V=\frac{1}{2}ax^2-mgx, \tag{34}$$

and the dissipation function is:

$$F=\frac{1}{2}R\dot{q}^2. \tag{35}$$

With this the Lagrangian for the system is

$$L=\frac{1}{2}m\dot{x}^2+\frac{1}{2}L_1(x)\dot{q}^2-\frac{1}{2}ax^2+mgx\,, \tag{36}$$

where x, $\dot{x}$ and $\dot{q}$ are scalars and the configuration space Q is 2-dimensional. The discrete Lagrangian of the system is

$$\begin{aligned}L_{\rm d}(x_{k-1},x_k,q_{l-1},q_l)=&\frac{h}{2}m\Big(\frac{x_k-x_{k-1}}{h}\Big)^2\\&+\frac{h}{2}\frac{L_1\big((1-\alpha)x_{k-1}+\alpha x_k\big)+L_1\big(\alpha x_{k-1}+(1-\alpha)x_k\big)}{2}\Big(\frac{q_l-q_{l-1}}{h}\Big)^2\\&-\frac{h}{2}a\frac{[(1-\alpha)x_{k-1}+\alpha x_k]^2+[\alpha x_{k-1}+(1-\alpha)x_k]^2}{2}+\frac{h}{2}mg(x_{k-1}+x_k)\,,\end{aligned} \tag{37}$$

the discrete dissipation function is given by

$$F_{\rm d}^-=\frac{1}{2}hR\Big(\frac{q_l-q_{l-1}}{h}\Big)^2,\qquad F_{\rm d}^+=\frac{1}{2}hR\Big(\frac{q_{l+1}-q_l}{h}\Big)^2, \tag{38}$$

such that

$$\frac{\partial F}{\partial\dot{q}}\Big|_{\rm d}^-=hR(q_l-q_{l-1})\,,\qquad \frac{\partial F}{\partial\dot{q}}\Big|_{\rm d}^+=-hR(q_{l+1}-q_l)\,, \tag{39}$$

and since in this case $Q''=0$ and $u=E$, we have

$$Q''_{\rm d}=0,\quad u_{\rm d}=hE. \tag{40}$$

Substituting Eqs.(37) to Eq.(40) into Eqs.(24) and dividing by h to restore the correct dimensionality, we obtain the discrete Euler–Lagrange equations which correspond to the equations in differences

$$\begin{aligned}&m\frac{x_{k+1}-2x_k+x_{k-1}}{h^2}-\frac{\alpha L_1'\big((1-\alpha)x_{k-1}+\alpha x_k\big)+(1-\alpha)L_1'\big(\alpha x_{k-1}+(1-\alpha)x_k\big)}{4}\Big(\frac{q_l-q_{l-1}}{h}\Big)^2\\&\quad-\frac{(1-\alpha)L_1'\big((1-\alpha)x_k+\alpha x_{k+1}\big)+\alpha L_1'\big(\alpha x_k+(1-\alpha)x_{k+1}\big)}{4}\Big(\frac{q_{l+1}-q_l}{h}\Big)^2\\&\quad+a\Big[\alpha(1-\alpha)x_{k+1}+\Big(\alpha^2+(1-\alpha)^2\Big)x_k+\alpha(1-\alpha)x_{k-1}\Big]+mgh=0\,,\\&\frac{1}{h}\frac{L_1\big((1-\alpha)x_k+\alpha x_{k+1}\big)+L_1\big(\alpha x_k+(1-\alpha)x_{k+1}\big)}{2}\Big(\frac{q_{l+1}-q_l}{h}\Big)\\&\quad-\frac{1}{h}\frac{L_1\big((1-\alpha)x_{k-1}+\alpha x_k\big)+L_1\big(\alpha x_{k-1}+(1-\alpha)x_k\big)}{2}\Big(\frac{q_l-q_{l-1}}{h}\Big)+2R\frac{q_{l+1}-q_{l-1}}{2h}-2E=0\,,\end{aligned} \tag{41}$$

where L_1' represents the derivative of L_1 with respect to its argument. If the discrete Lagrangian (37) is generalized difference-invariant under the infinitesimal transformation (26), and Eqs.(41) hold, the discrete Lagrange–Maxwell mechanico-electrical system has a first integral of the form

$$\begin{aligned}&\Big[-m\frac{x_k-x_{k-1}}{h}+\frac{h}{2}\frac{(1-\alpha)L_1'\big((1-\alpha)x_{k-1}+\alpha x_k\big)+\alpha L_1'\big(\alpha x_{k-1}+(1-\alpha)x_k\big)}{2}\\&\quad\times\Big(\frac{q_l-q_{l-1}}{h}\Big)^2-ha\frac{((1-\alpha)^2+\alpha^2)x_{k-1}+2\alpha(1-\alpha)x_k}{2}+\frac{h}{2}mg\Big]\xi_k\end{aligned}$$

$$-\frac{L_1\big((1-\alpha)x_{k-1}+\alpha x_k\big)+L_1\big(\alpha x_{k-1}+(1-\alpha)x_k\big)}{2}\,\frac{q_l-q_{l-1}}{h}\,\eta_l+\nu(x_k,x_{k-1},q_l,q_{l-1})=\text{const.} \tag{42}$$

Once $L_1(x)$ is known, we can build the function $v(x_k, x_{k-1}, q_l, q_{l-1})$ for some specific infinitesimal transformations. For instance, if we choose $L_1(x) = \text{const}$, and the infinitesimal transformations $\xi = 1$, $\eta = 1$, i.e.

$$x^* = x + \varepsilon\,, \qquad q^* = q + \varepsilon\,, \tag{43}$$

we have: $v = Rq_{k-1} + Eh$, and the system possesses the first integral

$$-m\frac{x_k-x_{k-1}}{h}-ha\frac{\Big((1-\alpha)^2+\alpha^2\Big)x_{k-1}+2\alpha(1-\alpha)x_k}{2}+hmg-L_1\frac{q_l-q_{l-1}}{h}+Rq_{l-1}+Eh=\text{const.} \tag{44}$$

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