ELSEVIER

Non-Noether symmetries and Lutzky conserved quantities for mechanico-electrical systems

Jing-Li Fu [a,c,*], Li-Qun Chen [b,c], Salvador Jiménez [d], Yi-Fa Tang [e]

[a] *Department of Physics, Shangqiu Normal University, Shangqiu 476000, China*
[b] *Department of Mechanics, Shanghai University, Shanghai 200444, China*
[c] *Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China*
[d] *Departamento de Matemática Aplicada TTII, E.T.S.I. Telecomunicación, Universidad Politécnica de Madrid, 28040 Madrid, Spain*
[e] *State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences, Beijing 100080, China*

Abstract

We apply the non-Noether symmetry theory for mechanical systems to Lagrange–Maxwell mechanico-electrical systems. For these systems, we derive the Lutzky conserved quantities from the corresponding equations of motion, the non-conservative and the dissipative forces, and the Lagrangian. Also, a condition that characterizes when a non-Noether symmetry leads to a Noether conservation law is presented.
© 2006 Elsevier B.V. All rights reserved.

1. Introduction

Symmetries play a key role in mathematics, physics and very specially in mechanics. The study of those quantities which are conserved by a mechanical system is highly relevant both from a theoretical and from a practical point of view. Different methods for finding such conserved quantities are known, those based on Noether theory [1], which addresses the invariance of the action functional under infinitesimal transformations, have proved powerful and widely used. Several extensions to this theory have been developed, that by Djanic and Vujanovic [2] to non-conservative holonomic systems via generalized velocities being specially worth mentioning. Further, Li [3] built a generalized Noether theory for non-linear non-holonomic dynamical systems, which has found wide application in the literature [4–7]. More recently, Crâşmăreanu [8] has constructed a Noether symmetry for 2D symmetry spinning particle.

A different approach to the finding of conserved quantities is based on those Lie symmetries which leave the equations of motion invariant under infinitesimal transformations, but do not necessarily do so with the action. These so-called non-Noether conserved quantities have proven of central importance in the study of dynamical systems. Thus, for Lagrangian systems, Lutzky has given such quantities (so-called Lutzky conserved quantities) both for Lie point symmetries and for velocity-dependent symmetries, none of them leaving the action invariant, and moreover has pointed out how these symmetries may lead to Noether-like symmetries, whenever they leave the action invariant [10–13]. Cicogna and Gaeta [14] have further studied Lie point symmetries in mechanics, and have obtained some conditions on their existence. Moreover, Hojman [15] has proven a theorem which may be used to construct some non-Noether conserved quantities (named after Hojman) directly from Lie symmetries, without resorting to Lagrangian or Hamiltonian functions. Further, Mei [16] has made major progress in the study of Lie symmetries for constrained mechanical systems, though restricted to Noether conserved quan-

---

tities. More recently, some of the authors [17] have made a contribution to the study of non-Noether symmetries for non-conservative dynamical systems.

In this Letter, we study the motion of a mechanico-electrical dynamical system, under both conservative and non-conservative, and also dissipative forces. For this system, we have extended the study of the symmetries and derived an expression for the Lutzky conserved quantities. Also, we have estabished the conditions under which non-Noether symmetries result in Noether symmetries.

2. Lagrange–Maxwell equations for mechanico-electrical systems

A mechanico-electrical dynamical system couples a mechanical process to an electromagnetic process. The mechanical part consists of N particles, described by n generalized coordinates q_s $(s = 1, \ldots, n)$. The electrodynamical part corresponds to m electrical circuits consisting of linear conductors and capacitors. For circuit k, we denote by i_k the current, by u_k the electric potential, by e_k the capacitor charge (with $\dot{e}_k = i_k$), by R_k the resistance and by C_k the capacitance.

The Lagrangian for such a mechanico-electrical system is

$$L = T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}) + W_{\mathrm{m}}(\mathbf{q}, \dot{\mathbf{e}}) - W_{\mathrm{e}}(\mathbf{q}, \mathbf{e}), \tag{1}$$

where T and V are, respectively, the kinetic and the potential energy. The electric field energy and the magnetic field energy are, respectively, defined by

$$W_{\mathrm{e}} = \sum_{k=1}^{m} \frac{1}{2} \frac{e_k^2}{C_k}, \qquad W_{\mathrm{m}} = \sum_{k=1}^{m} \sum_{r=1}^{m} \frac{1}{2} L_{kr} i_k i_r, \tag{2}$$

where L_{kr} (with $k \neq r$) is the inductance on circuit k due to circuit r and L_{kk} is the self-inductance of circuit k.

The motion of the system is given by the Lagrange–Maxwell system of equations [18,19]:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s} = Q_s \quad (s = 1, \ldots, n),$$
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} + \frac{\partial F}{\partial \dot{e}_k} = u_k \quad (k = 1, \ldots, m), \tag{3}$$

where Q_s $(s = 1, \ldots, n)$ are generalized, non-conservative forces. This is a system of $n + m$ ordinary differential equations with respect to the n generalized coordinates q_s $(s = 1, \ldots, n)$ and the m generalized electric quantities e_k $(k = 1, \ldots, m)$. The dissipative function F of the system is given by

$$F = F_{\mathrm{e}}(\dot{\mathbf{e}}) + F_{\mathrm{m}}(\mathbf{q}, \dot{\mathbf{q}}), \tag{4}$$

where the electric dissipation function is

$$F_{\mathrm{e}} = \frac{1}{2} \sum_{k=1}^{m} R_k i_k^2 = \frac{1}{2} \sum_{k=1}^{m} R_k \dot{e}_k^2, \tag{5}$$

and F_{m} is the function for the viscous frictional damping forces, and thus, $-\partial F/\partial \dot{q}_s$ $(s = 1, \ldots, n)$ correspond to dissipative forces.

When the mechanico-electrical system satisfies the conditions $Q_s - \partial F/\partial \dot{q}_s = 0$ $(s = 1, \ldots, n)$ and $u_k - \partial F/\partial \dot{e}_k = 0$

$(k = 1, \ldots, m)$, system (3) reduces to a set of Lagrange equations of the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \quad (s = 1, \ldots, n),$$
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} = 0 \quad (k = 1, \ldots, m). \tag{6}$$

In such a case, the system corresponds to a Lagrangian mechanico-electrical system.

3. Non-Noether symmetries and Lutzky conserved quantities for Lagrange–Maxwell mechanico-electrical systems

The system of Lagrange–Maxwell equations (3) for the mechanico-electrical system can be written in compact form as

$$\ddot{q}_s = \alpha_s(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}, t) \quad (s = 1, \ldots, n),$$
$$\ddot{e}_k = \beta_k(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}, t) \quad (k = 1, \ldots, m). \tag{7}$$

Let us introduce infinitesimal transformations with respect to the generalized coordinates, electric charges and time:

$$t^* = t + \varepsilon \xi_0(\mathbf{q}, \mathbf{e}, t), \qquad q_s^* = q_s + \varepsilon \xi_s(\mathbf{q}, \mathbf{e}, t),$$
$$e_k^* = e_k + \varepsilon \eta_k(\mathbf{q}, \mathbf{e}, t), \tag{8}$$

where ε is a small parameter, and ξ_0, ξ_s and η_k $(s = 1, \ldots, n;$ $k = 1, \ldots, m)$ are the corresponding infinitesimal generators. To assume system (7) invariant under the infinitesimal transformations (8) leads to the determining equations

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\alpha_s \dot{\xi}_0 = X^{(1)}(\alpha_s) \quad (s = 1, \ldots, n),$$
$$\ddot{\eta}_k - \dot{e}_k \ddot{\xi}_0 - 2\beta_k \dot{\xi}_0 = X^{(1)}(\beta_k) \quad (k = 1, \ldots, m). \tag{9}$$

From here onwards, we will use the convention of summation over repeated indexes. Operator $X^{(1)}$ is the generator of the first extended group [9], and is given by

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \eta_k \frac{\partial}{\partial e_k} + (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s}$$
$$+ (\dot{\eta}_k - \dot{e}_k \dot{\xi}_0) \frac{\partial}{\partial \dot{e}_k}, \tag{10}$$

and the vector field

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \dot{e}_k \frac{\partial}{\partial e_k} + \alpha_s \frac{\partial}{\partial \dot{q}_s} + \beta_k \frac{\partial}{\partial \dot{e}_k}, \tag{11}$$

supposes derivation with respect to time along the trajectories of the system of equations (7). Thus, for any function ϕ, we have

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \dot{q}_s \frac{\partial \phi}{\partial q_s} + \alpha_s \frac{\partial \phi}{\partial \dot{q}_s} + \dot{e}_k \frac{\partial \phi}{\partial e_k} + \beta_k \frac{\partial \phi}{\partial \dot{e}_k}. \tag{12}$$

Whenever the infinitesimal transformation (8) leave equations (7) invariant but do not leave the action invariant, we have a non-Noether symmetry for the Lagrange–Maxwell mechanico-electrical system. Eqs. (9) can be regarded as a criterion for non-Noether symmetries:

Criterion 1. The Lagrange–Maxwell mechanico-electrical system given by Eqs. (7) possesses a non-Noether symmetry if the infinitesimal generators ξ_0, ξ_s and η_k satisfy the determining equations (9) and do not leave the action of the system invariant.

It can be easily seen [17] that the relations between the terms α_s, β_k, $Q_s - \partial F/\partial \dot{q}_s$, $u_k - \partial F/\partial \dot{e}_k$ and the Lagrangian L are given by

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} - \frac{\partial}{\partial \dot{q}_l}\left(\frac{M_{ls}}{D_1}\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\right) + \frac{d}{dt}(\ln D_1) = 0$$
$$(s, l = 1, \ldots, n),$$
$$\frac{\partial \beta_k}{\partial \dot{e}_k} - \frac{\partial}{\partial \dot{e}_j}\left(\frac{N_{jk}}{D_2}\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\right) + \frac{d}{dt}(\ln D_2) = 0$$
$$(k, j = 1, \ldots, m), \tag{13}$$

where

$$D_1 = \det[\partial^2 L/\partial \dot{q}_s \partial \dot{q}_l], \qquad D_2 = \det[\partial^2 L/\partial \dot{e}_k \partial \dot{e}_j], \tag{14}$$

and M_{ls} and N_{jk} are, respectively, the cofactor of $\partial^2 L/\partial \dot{q}_s \partial \dot{q}_l$ and of $\partial^2 L/\partial \dot{e}_k \partial \dot{e}_j$ for the matrices given by the second derivatives.

Theorem 1. *The Lagrange–Maxwell mechanico-electrical system given by* (7) *possesses a Lutzky conserved quantity of the form*

$$\Phi = 2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}\right) + 2\left(\frac{\partial \eta_k}{\partial e_k} - \dot{e}_k\frac{\partial \xi_0}{\partial e_k}\right) - N\dot{\xi}_0$$
$$+ X^{(1)}(\ln D_1 D_2) - \int\left[X^{(1)}\left[\frac{\partial}{\partial \dot{q}_s}\left(\frac{M_{ls}}{D_1}\left(Q_l - \frac{\partial F}{\partial \dot{q}_l}\right)\right)\right.\right.$$
$$\left.+ \frac{\partial}{\partial \dot{e}_k}\left(\frac{N_{jk}}{D_2}\left(u_j - \frac{\partial F}{\partial \dot{e}_j}\right)\right)\right]$$
$$- \dot{\xi}_0\frac{\partial}{\partial \dot{q}_s}\left(\frac{M_{ls}}{D_1}\left(Q_l - \frac{\partial F}{\partial \dot{q}_l}\right)\right)$$
$$\left.- \dot{\xi}_0\frac{\partial}{\partial \dot{e}_k}\left(\frac{N_{jk}}{D_2}\left(u_j - \frac{\partial F}{\partial \dot{e}_j}\right)\right)\right]dt, \tag{15}$$

if the generators ξ_0, ξ_s and η_k satisfy the determining equations (9), *and do not leave invariant the action of the system.*

Proof. Let us pass all the terms in Eqs. (9) to the left-hand side, and denote the resulting expressions by Π_s and Π_k. Their partial derivatives with respect to $\dot{q}_s$ and $\dot{e}_k$ are respectively

$$\frac{\partial \Pi_s}{\partial \dot{q}_s} = \frac{d}{dt}\left[2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}\right) - n\dot{\xi}_0\right]$$
$$- X^{(1)}\left(\frac{\partial \alpha_s}{\partial \dot{q}_s}\right) - \frac{\partial \alpha_s}{\partial \dot{q}_s}\dot{\xi}_0,$$
$$\frac{\partial \Pi_k}{\partial \dot{e}_k} = \frac{d}{dt}\left[2\left(\frac{\partial \eta_k}{\partial e_k} - \dot{e}_k\frac{\partial \xi_0}{\partial e_k}\right) - m\dot{\xi}_0\right]$$
$$- X^{(1)}\left(\frac{\partial \beta_k}{\partial \dot{e}_k}\right) - \frac{\partial \beta_k}{\partial \dot{e}_k}\dot{\xi}_0. \tag{16}$$

If ξ_0, ξ_s and η_k satisfy Eqs. (9), then, from Eqs. (13) and (14), we have:

$$-X^{(1)}\frac{\partial \alpha_s}{\partial \dot{q}_s} - \dot{\xi}_0\frac{\partial \alpha_s}{\partial \dot{q}_s}$$
$$= \frac{d}{dt}X^{(1)}(\ln D_1) - X^{(1)}\frac{\partial}{\partial \dot{q}_l}\left[\frac{M_{ls}}{D_1}\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\right]$$
$$+ \dot{\xi}_0\frac{\partial}{\partial \dot{q}_l}\left[\frac{M_{ls}}{D_1}\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\right],$$
$$-X^{(1)}\frac{\partial \beta_k}{\partial \dot{e}_k} - \dot{\xi}_0\frac{\partial \beta_k}{\partial \dot{e}_k}$$
$$= \frac{d}{dt}X^{(1)}(\ln D_2) - X^{(1)}\frac{\partial}{\partial \dot{e}_j}\left[\frac{N_{jk}}{D_2}\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\right]$$
$$+ \dot{\xi}_0\frac{\partial}{\partial \dot{e}_j}\left[\frac{N_{jk}}{D_2}\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\right]. \tag{17}$$

With this, we may express (16) in the form

$$\frac{\partial \Pi_s}{\partial \dot{q}_s} = \frac{d}{dt}\left[2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}\right) - n\dot{\xi}_0 + X^{(1)}(\ln D_1)\right]$$
$$- X^{(1)}\left[\frac{\partial}{\partial \dot{q}_l}\left(\frac{M_{ls}}{D_1}\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\right)\right]$$
$$- \dot{\xi}_0\frac{\partial}{\partial \dot{q}_l}\left(\frac{M_{ls}}{D_1}\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\right),$$
$$\frac{\partial \Pi_k}{\partial \dot{e}_k} = \frac{d}{dt}\left[2\left(\frac{\partial \eta_k}{\partial e_k} - \dot{e}_k\frac{\partial \xi_0}{\partial e_k}\right) - m\dot{\xi}_0 + X^{(1)}(\ln D_2)\right]$$
$$- X^{(1)}\left[\frac{\partial}{\partial \dot{e}_j}\left(\frac{N_{jk}}{D_2}\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\right)\right]$$
$$- \dot{\xi}_0\frac{\partial}{\partial \dot{e}_j}\left(\frac{N_{jk}}{D_2}\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\right). \tag{18}$$

Furthermore, if ξ_0, ξ_s and η_k satisfy both $\Pi_s = \Pi_k = 0$ and $\partial \Pi_s/\partial \dot{q}_s = \partial \Pi_k/\partial \dot{e}_k = 0$, $s = 1, \ldots, n$; $k = 1, \ldots, m$, then Eqs. (18) imply that Φ, as given by Eq. (15), is a conserved quantity. $\square$

This is the main result in the present Letter. It is important to stress that even though it was necessary to postulate that the equations of the motion were derived from a Lagrangian L, with generalized forces Q_s and electric potentials u_k in order to arrive at the conserved quantity (15), we did not assume that the symmetry group leaves the action invariant.

From this, we can give the following significant result:

Theorem 2. *If the infinitesimal-transformation generators ξ_0, ξ_s and η_k $(s = 1, \ldots, n; k = 1, \ldots, m)$ satisfy Eqs.* (9), *then the Lagrange–Maxwell mechanico-electrical system given by* (7) *has a Lutzky conserved quantity of the form*

$$\Phi = 2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}\right) + 2\left(\frac{\partial \eta_k}{\partial e_k} - \dot{e}_k\frac{\partial \xi_0}{\partial e_k}\right) - (n+m)\dot{\xi}_0$$
$$+ X^{(1)}(\ln(D_1 D_2)) - X^{(1)}(f) \tag{19}$$

if and only if there exists a function $f = f(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}, t) = f_1(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}, t) + f_2(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}, t)$ that satisfies

$$\frac{df_1}{dt} = \frac{\partial}{\partial \dot{q}_l}\left(\frac{M_{ls}}{D_1}\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\right),$$

$$\frac{df_2}{dt} = \frac{\partial}{\partial \dot{e}_j}\left(\frac{N_{jk}}{D_2}\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\right). \tag{20}$$

From Theorem 2, we can obtain a Lutzky conserved quantity associated to a mechanico-electrical system, and the system of equations (20) can be understood as a restrictive condition that corresponds to generalized forces.

4. Non-Noether symmetries and Lutzky conserved quantities for a Lagrangian mechanico-electrical system

Let us suppose that the mechanico-electrical system is now a Lagrangian one, and that the equations of motion can be expressed in the form

$$\ddot{q}_s = \alpha'_s(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}) \quad (s = 1, \ldots, n),$$
$$\ddot{e}_k = \beta'_k(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}) \quad (k = 1, \ldots, m). \tag{21}$$

Supposing that the equations of motion (21) are invariant under the infinitesimal transformations (8), we obtain the corresponding set of determining equations for this case:

$$\ddot{\xi}_s - \dot{q}_s\ddot{\xi}_0 - 2\alpha'_s\dot{\xi}_0 = X^{(1)}(\alpha'_s), \quad s = 1, \ldots, n,$$
$$\ddot{\eta}_k - \dot{e}_k\ddot{\xi}_0 - 2\beta'_k\dot{\xi}_0 = X^{(1)}(\beta'_k), \quad k = 1, \ldots, m, \tag{22}$$

where operator $X^{(1)}$ is the generator of the first extended group.

We may consider the system of equations (22) as a criterion for non-Noether symmetries:

Criterion 2. The Lagrangian mechanico-electrical system given by Eqs. (21) possesses a non-Noether symmetry if the infinitesimal generators $\xi_0(\mathbf{q}, \mathbf{e}, t)$, $\xi_s(\mathbf{q}, \mathbf{e}, t)$ and $\eta_k(\mathbf{q}, \mathbf{e}, t)$ $(s = 1, \ldots, n; k = 1, \ldots, m)$ satisfy the determining equations (22) and do not leave the action of the system invariant.

To obtain the Lutzky conserved quantities, we need two results. On one hand, since the equations of motion (21) are in this case of the form (6), we may rewrite them equivalently as:

$$\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_l}\ddot{q}_l = \frac{\partial L}{\partial q_s} - \frac{\partial^2 L}{\partial \dot{q}_s \partial t} - \frac{\partial^2 L}{\partial \dot{q}_s \partial q_l}\dot{q}_l,$$
$$\frac{\partial^2 L}{\partial \dot{e}_k \partial \dot{e}_j}\ddot{e}_j = \frac{\partial L}{\partial e_k} - \frac{\partial^2 L}{\partial \dot{e}_k \partial t} - \frac{\partial^2 L}{\partial \dot{e}_k \partial e_j}\dot{e}_j, \tag{23}$$

and we can deduce the following relations between the α'_s, the β'_k and the Lagrangian L:

$$\frac{\partial \alpha'_s}{\partial \dot{q}_s} + \frac{d}{dt}(\ln D_1) = 0, \qquad \frac{\partial \beta'_k}{\partial \dot{e}_k} + \frac{d}{dt}(\ln D_2) = 0, \tag{24}$$

with

$$D_1 = \det\left[\partial^2 L/\partial \dot{q}_s \partial \dot{q}_l\right] \quad (s, l = 1, \ldots, n),$$
$$D_2 = \det\left[\partial^2 L/\partial \dot{e}_k \partial \dot{e}_j\right] \quad (k, j = 1, \ldots, m). \tag{25}$$

On the other hand, we remark that if ξ_0, ξ_s and η_k $(s = 1, \ldots, n; k = 1, \ldots, m)$ satisfy the system of equations (22), it can be shown [10] that

$$\dot{X}^{(1)}(\phi) = X^{(1)}(\dot{\phi}) + \dot{\xi}_0\dot{\phi} \tag{26}$$

holds for any function $\phi(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{e}, \dot{\mathbf{e}}, t)$. With these two results, one can prove

Theorem 3. *The Lagrangian mechanico-electrical system given by* (21) *possesses a Lutzky conserved quantity of the form*

$$\Phi = 2\left(\frac{\partial \xi_s}{\partial q_s} - \dot{q}_s\frac{\partial \xi_0}{\partial q_s}\right) + 2\left(\frac{\partial \eta_k}{\partial e_k} - \dot{e}_k\frac{\partial \xi_0}{\partial e_k}\right) - (n+m)\dot{\xi}_0$$
$$+ X^{(1)}\left(\ln(D_1 D_2)\right), \tag{27}$$

if the infinitesimal-transformation generators ξ_0, ξ_s and η_k ($s = 1, \ldots, n; k = 1, \ldots, m$) satisfy the determining equations (22), *and do not leave invariant the action of the system.*

Theorem 3 can be proven in a similar way to Theorem 1. With this result, we characterize the conserved quantity Φ. It should be noted, once again, that it is necessary to assume the equations of the motion being derived from a Lagrangian, but that it is not necessary to assume the action being invariant.

5. Noether symmetry derived from a non-Noether symmetry for a mechanico-electrical system

We will now consider the case when the symmetry does preserve the action of the system. The main result is

Theorem 4. *If the Lagrange–Maxwell mechanico-electrical system possesses a Lutzky conserved quantity of the form*

$$\tilde{\Phi} = -\frac{\partial^2}{\partial \dot{q}_l \partial \dot{q}_s}\left[\frac{M_{ls}}{D_1}(\xi_s - \dot{q}_s\xi_0)\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\right]$$
$$- \frac{\partial^2}{\partial \dot{e}_j \partial \dot{e}_k}\left[\frac{N_{jk}}{D_2}(\eta_k - \dot{e}_k\xi_0)\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\right]$$
$$- \int\left(X^{(1)} - \dot{\xi}_0\right)\left[\frac{\partial}{\partial \dot{q}_s}\left(\frac{M_{ls}}{D_1}\left(Q_l - \frac{\partial F}{\partial \dot{q}_l}\right)\right)\right.$$
$$+ \frac{\partial}{\partial \dot{e}_k}\left(\frac{N_{jk}}{D_2}\left(u_j - \frac{\partial F}{\partial \dot{e}_j}\right)\right)\right]dt, \tag{28}$$

then, the symmetry transformation group given by ξ_0, ξ_s and η_k leaves invariant the action of the system, and the non-Noether symmetry leads to a Noether symmetry with a Noether conserved quantity of the form

$$I = \xi_0 L + (\xi_s - \dot{q}_s\xi_0)\frac{\partial L}{\partial \dot{q}_s} + (\eta_k - \dot{e}_k\xi_0)\frac{\partial L}{\partial \dot{e}_k} + G_N$$
$$= \text{const}, \tag{29}$$

whenever a gauge function G_N can be found.

Proof. We can verify by direct calculation that for any functions ξ_0, ξ_s and η_k we have

$$X^{(1)}\left(\frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s}\right) = \frac{\partial^2 X^{(1)}(L)}{\partial \dot{q}_l \partial \dot{q}_s} - \frac{\partial L}{\partial \dot{q}_r}\frac{\partial(A_{rs})}{\partial \dot{q}_l}$$
$$- A_{rs}\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_l} - A_{rl}\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s},$$

$$X^{(1)}\left(\frac{\partial^2 L}{\partial \dot{e}_j \partial \dot{e}_k}\right) = \frac{\partial^2 X^{(1)}(L)}{\partial \dot{e}_j \partial \dot{e}_k} - \frac{\partial L}{\partial \dot{e}_p}\frac{\partial(C_{pk})}{\partial \dot{e}_j}$$

$$- C_{pk}\frac{\partial^2 L}{\partial \dot{e}_p \partial \dot{e}_j} - C_{pj}\frac{\partial^2 L}{\partial \dot{e}_p \partial \dot{e}_k},$$

$$(s, l, r = 1, \ldots, n; \ k, j, p = 1, \ldots, m), \tag{30}$$

where

$$A_{rl} = \frac{\partial \xi_r}{\partial q_l} - \dot{q}_r\frac{\partial \xi_0}{\partial q_l} - \dot{\xi}_0\delta_{rl},$$

$$C_{pj} = \frac{\partial \eta_p}{\partial e_j} - \dot{e}_p\frac{\partial \xi_0}{\partial e_j} - \dot{\xi}_0\delta_{pj}. \tag{31}$$

Noether's theorem states [7] that if ξ_0, ξ_s and η_k generate a Noether symmetry corresponding to the Lagrange–Maxwell mechanico-electrical system, there exists a function $G_N(t, \mathbf{q}, \mathbf{e})$ such that

$$X^{(1)}(L) + \dot{\xi}_0 L + (\xi_r - \dot{q}_r\xi_0)\left(Q_r - \frac{\partial F}{\partial \dot{q}_r}\right)$$

$$+ (\eta_p - \dot{e}_p\xi_0)\left(u_p - \frac{\partial F}{\partial \dot{e}_p}\right) = -\dot{G}_N. \tag{32}$$

Since the right-hand side of Eq. (32) is linear in terms of the generalized velocities and electric currents, we have

$$\frac{\partial^2 X^{(1)}(L)}{\partial \dot{q}_l \partial \dot{q}_s}$$

$$= -\frac{\partial^2(\dot{\xi}_0 L)}{\partial \dot{q}_l \partial \dot{q}_s} - \frac{\partial^2}{\partial \dot{q}_l \partial \dot{q}_s}\left[(\xi_r - \dot{q}_r\xi_0)\left(Q_r - \frac{\partial F}{\partial \dot{q}_r}\right)\right],$$

$$\frac{\partial^2 X^{(1)}(L)}{\partial \dot{e}_j \partial \dot{e}_k}$$

$$= -\frac{\partial^2(\dot{\xi}_0 L)}{\partial \dot{e}_j \partial \dot{e}_k} - \frac{\partial^2}{\partial \dot{e}_j \partial \dot{e}_k}\left[(\eta_p - \dot{e}_p\xi_0)\left(u_p - \frac{\partial F}{\partial \dot{e}_p}\right)\right], \tag{33}$$

where we have used that

$$\frac{\partial^2}{\partial \dot{q}_l \partial \dot{q}_s}\left[(\eta_p - \dot{e}_p\xi_0)\left(u_p - \frac{\partial F}{\partial \dot{e}_p}\right)\right] = 0,$$

$$\frac{\partial^2}{\partial \dot{e}_j \partial \dot{e}_k}\left[(\xi_r - \dot{q}_r\xi_0)\left(Q_r - \frac{\partial F}{\partial \dot{q}_r}\right)\right] = 0.$$

Substituting Eqs. (33) into Eqs. (30), we obtain

$$X^{(1)}\left(\frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s}\right) = \dot{\xi}_0\frac{\partial^2 L}{\partial \dot{q}_l \partial \dot{q}_s} - B_{rs}\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_l} - B_{rl}\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s}$$

$$- \frac{\partial^2}{\partial \dot{q}_l \partial \dot{q}_s}\left[(\xi_r - \dot{q}_r\xi_0)\left(Q_r - \frac{\partial F}{\partial \dot{q}_r}\right)\right],$$

$$X^{(1)}\left(\frac{\partial^2 L}{\partial \dot{e}_j \partial \dot{e}_k}\right) = \dot{\xi}_0\frac{\partial^2 L}{\partial \dot{e}_j \partial \dot{e}_k} - D_{pk}\frac{\partial^2 L}{\partial \dot{e}_p \partial \dot{e}_j} - D_{pj}\frac{\partial^2 L}{\partial \dot{e}_p \partial \dot{e}_k}$$

$$- \frac{\partial^2}{\partial \dot{e}_j \partial \dot{e}_k}\left[(\eta_p - \dot{e}_p\xi_0)\left(u_p - \frac{\partial F}{\partial \dot{e}_p}\right)\right], \tag{34}$$

where

$$B_{ls} = \frac{\partial \xi_l}{\partial q_s} - \dot{q}_l\frac{\partial \xi_0}{\partial q_s}, \qquad D_{jk} = \frac{\partial \eta_j}{\partial e_k} - \dot{e}_j\frac{\partial \xi_0}{\partial e_k}. \tag{35}$$

Let M_{ts} and N_{vk} be, respectively, the cofactors of the elements $\frac{\partial^2 L}{\partial \dot{q}_t \partial \dot{q}_s}$ and $\frac{\partial^2 L}{\partial \dot{e}_v \partial \dot{e}_k}$ of the matrices formed by these second derivatives. From the properties of determinants, we have

$$M_{ts}\frac{\partial^2 L}{\partial \dot{q}_t \partial \dot{q}_r} = D_1\delta_{sr}, \qquad N_{vk}\frac{\partial^2 L}{\partial \dot{e}_v \partial \dot{e}_p} = D_2\delta_{kp}, \tag{36}$$

and

$$M_{ts}\frac{\partial}{\partial \rho}\frac{\partial^2 L}{\partial \dot{q}_t \partial \dot{q}_s} = \frac{\partial D_1}{\partial \rho}, \qquad N_{vk}\frac{\partial}{\partial \gamma}\frac{\partial^2 L}{\partial \dot{e}_v \partial \dot{e}_k} = \frac{\partial D_2}{\partial \gamma}, \tag{37}$$

with D_1 and D_2 given by (14), and where ρ stands for any of q_s, $\dot{q}_s$ and t, and γ stands for any of e_k, $\dot{e}_k$ and t.

Multiplying each of the two equations in (34), respectively, by M_{ts} and N_{vk}, summing on the repeated indexes, and using Eqs. (36) and (37), we have

$$X^{(1)}(\ln D_1)$$

$$= n\dot{\xi}_0 - 2B_{rr} - \frac{\partial^2}{\partial \dot{q}_l \partial \dot{q}_s}\left[\frac{M_{ls}}{D_1}(\xi_s - \dot{q}_s\xi_0)\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)\right],$$

$$X^{(1)}(\ln D_2)$$

$$= m\dot{\xi}_0 - 2D_{pp} - \frac{\partial^2}{\partial \dot{e}_j \partial \dot{e}_k}\left[\frac{N_{jk}}{D_2}(\eta_k - \dot{e}_k\xi_0)\left(u_k - \frac{\partial F}{\partial \dot{e}_k}\right)\right]. \tag{38}$$

From Eqs. (38) and (15), one can prove the conservation of Φ whenever the symmetry group leaves the action invariant. $\quad\Box$

In this case, however, we have the classical Noether invariance result for the Lagrange–Maxwell mechanico-electrical system. According to Lutzky's ideas [11], we may conjecture that the Lutzky conserved quantities (15) and (27) need not be "new" conserved quantities. That is, they may be represented in terms of the Noether invariants of the system. In fact, Noether invariants form a complete set of conserved quantities; for the mechanico-electrical system, any additional constant of the motion must necessarily be a function of the Noether invariants.

6. Conclusion

In this Letter, we have extended non-Noether symmetries to Lagrangian and Lagrange–Maxwell mechanico-electrical systems with mechanico-electrical coupling and dissipation functions. Our results represent a significant approach to finding conserved quantities for these systems.

Acknowledgements

References

[1] A.E. Noether, Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II (1918) 235.

[2] Dj.D. Djukic, B. Vujanovic, Acta Mech. 23 (1975) 17.

[3] Z.P. Li, Acta Phys. Sinica 30 (1981) 1659.

[4] D. Liu, Chin. Sci. (Ser. A) 20 (1990) 1189.

[5] F.X. Mei, Chin. Sci. (Ser. A) 23 (1993) 709.

[6] J.L. Fu, L.Q. Chen, Mech. Res. Commun. 31 (2004) 9.

[7] J.L. Fu, L.Q. Chen, Phys. Lett. A 331 (2004) 138.

[8] M. Crâşmăreanu, Int. J. Non-Linear Mech. 35 (2000) 947.

[9] M. Lutzky, J. Phys. A: Math. Gen. 12 (1979) 973.

[10] M. Lutzky, Phys. Lett. A 75 (1979) 8.

[11] M. Lutzky, Phys. Lett. A 72 (1979) 86.

[12] M. Lutzky, J. Phys. A: Math. Gen. 28 (1995) 637.

[13] M. Lutzky, Int. J. Non-Linear Mech. 33 (1998) 393.

[14] G. Cicogna, G. Gaeta, Nuovo Cimento B 107 (1992) 1085.

[15] S. Hojman, J. Phys. A: Math. Gen. 25 (1992) L291.

[16] F.X. Mei, Application of Lie Groups and Lie Algebra to Constraint Mechanical Systems, Science Press, Beijing, 1999 (in Chinese).

[17] J.L. Fu, L.Q. Chen, Phys. Lett. A 317 (2003) 255.

[18] J.J. Qiu, Dynamical Analysis of Mechanico-Electrical Systems, Science Press, Beijing, 1992.

[19] Y.Z. Liu, Advance Dynamics, Higher Education Press, Beijing, 2001.