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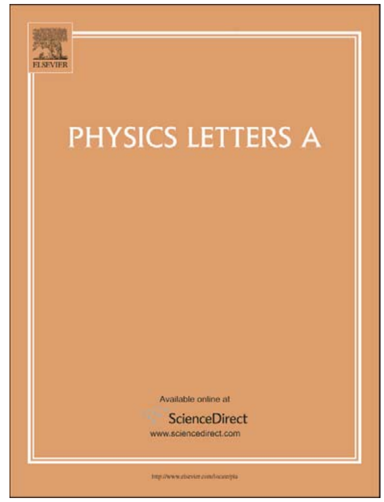

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Construction of exact invariants of time-dependent linear nonholonomic dynamical systems *

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Abstract

In this work, we build exact dynamical invariants for time-dependent, linear, nonholonomic Hamiltonian systems in two dimensions. Our aim is to obtain an additional insight into the theoretical understanding of generalized Hamilton canonical equations. In particular, we investigate systems represented by a quadratic Hamiltonian subject to linear nonholonomic constraints. We use a Lie algebraic method on the systems to build the invariants. The role and scope of these invariants is pointed out.

Keywords: invariant, Lie algebraic method, linear nonholonomic Hamiltonian system.

1 Introduction

Invariants, provided they exist and can be computed, are a very useful tool to understand the theoretical structure of dynamical systems [1–3]. Different methods have been developed to build invariants of mechanical and physical systems, such as the Ermakov technique [4, 5], the symmetries approach [6–11]

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and the discrete symmetries approach [12–16], the dynamical algebraic method [17–19] and the algebraic structure and Poisson methods [20–23]. Of all the methods, those making use of the algebraic structure present the additional advantage of being extended in a straightforward way to the corresponding quantum mechanical systems [24, 25]. In recent year, the use of a Lie algebraic approach to build dynamical invariants has provided many interesting results [18,19,26–30]. Fu *et al.* have used the algebraic structure and Poisson methods to study relativistic Birkhoffian systems.

In this paper we will consider the study of two-dimensional systems described by a time-dependent Hamiltonian of the form

$$H = \frac{1}{2}(\alpha_1 p_1^2 + \alpha_2 p_2^2) + \frac{1}{2}(\beta_1 q_1^2 + \beta_2 q_2^2) + \beta_3 q_1 q_2 + \alpha_3 p_1 p_2 \,,$$

subject to linear nonholonomic constraints such as

$$\sum_{s=1}^{n} a_{\varepsilon+\beta,s}\dot{q}_s + a_{\varepsilon+\beta}(t, \boldsymbol{q}) = 0, \quad (\beta = 1, \cdots, g; \varepsilon = n - g)\,.$$

The Hamiltonian has a coupling term for the momenta p_1 and p_2, and the coefficients $\alpha_i, \beta_j, i = 1, 2, 3, j = 1, 2, 3$ are in general time-dependent. Using specifically the Lie algebraic approach [30], we carry out in what follows the construction of dynamical invariants in the case when H is time-dependent.

2 Time-dependent invariants for linear nonholonomical systems

2.1 Generalized Poisson condition

Let us consider a system described by n general coordinates $q_s(s = 1, \ldots, n)$, with its motion subject to a conservative force and restricted by g ideal linear nonholonomic constraints:

$$\sum_{s=1}^{n} a_{\varepsilon+\beta,s}\dot{q}_s + a_{\varepsilon+\beta}(t, \boldsymbol{q}) = 0, \quad (\beta = 1, \cdots, g; \varepsilon = n - g), \tag{1}$$

and the generic force is potential. The equations of motion can be written in the form [33]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \sum_{\beta=1}^{g} \delta_\beta a_{\varepsilon+\beta,s} \qquad (s = 1, \cdots, n), \tag{2}$$

where δ_β are the multipliyers of the constraints. Introducing the Hamiltonian and the generalized momenta

$$H = \sum_{s=1}^{n} p_s\dot{q}_s - L, \quad p_s = \frac{\partial L}{\partial \dot{q}_s}, \tag{3}$$

the generalized Hamilton canonicals equations for the system are

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + \sum_{\beta=1}^{g}(\delta_\beta)a_{\varepsilon+\beta,s}\,. \tag{4}$$

In this notation, δ_β is expressed in position-velocity as a function of $q_s, \dot{q}_s, t$, before integrating the equations (4), while (δ_β) represents δ_β expressed in position-momentum as a function of q_s, p_s, t. In this method, one expresses the two-dimensional time-dependent Hamiltonian in the form

$$H = \sum_i h_i(t)\Gamma_i(q_1, q_2, p_1, p_2)\,. \tag{5}$$

The coefficients $h_i(t)$ are time-dependent and Γ_i are the phase-space functions required in order to close the algebra with respect to the Poisson bracket:

$$[\Gamma_i, \Gamma_j] = \sum_k C_{ij}^k \Gamma_k \,, \tag{6}$$

with C_{ij}^k the structure constants of the Lie algebra. In the two-dimensional case, the Poisson bracket is defined by

$$[A, B] = \frac{\partial A}{\partial q_1}\frac{\partial B}{\partial p_1} - \frac{\partial A}{\partial p_1}\frac{\partial B}{\partial q_1} + \frac{\partial A}{\partial q_2}\frac{\partial B}{\partial p_2} - \frac{\partial A}{\partial p_2}\frac{\partial B}{\partial q_2} \,. \tag{7}$$

In order to close the algebra, additional functions Γ_i might be necessary. In that case, they would appear in (5) with null coefficients $h_l(t) = 0$. Since any invariant I that we seek must also be a member of the dynamical algebra, it can expressed in the form

$$I = \sum_m \lambda_m(t) \Gamma_m(q_1, q_2, p_1, p_2) \,, \tag{8}$$

and the necessary and sufficient condition under which $I(q_s, p_s, t)$ is a first integral of the linear non-holonomic system given by (1) and (2) is [33]:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + [I, H] + \sum_{\beta=1}^{g} (\delta_\beta) a_{\varepsilon+\beta,s} \frac{\partial I}{\partial p_s} = 0 \,. \tag{9}$$

Substituting (5) and (8) in (9), using (6) and simplifying, we obtain at a set of first-order, coupled differential equations of the form

$$\dot{\lambda}_k + \sum_i \left[\sum_j C_{ij}^k h_j(t) \right] \lambda_i + D_k = 0 \,. \tag{10}$$

It is possible to show that $\sum_{\beta=1}^{g} (\delta_\beta) a_{\varepsilon+\beta,s} \frac{\partial I}{\partial p_s}$ can be expressed in terms of the Γ_i in such a way that, in equation (10), the terms D_k satisfy

$$\sum_k D_k \Gamma_k = \sum_i \left[\sum_{\beta=1}^{g} (\delta_\beta) a_{\varepsilon+\beta,s} \frac{\partial \Gamma_i}{\partial p_s} \right] \lambda_i \,. \tag{11}$$

The set of equations can be solved for the λ_s, and substituting them in (8) we obtain the required invariant I.

2.2 An example

Let us consider a system with the following Lagrangian

$$L = \frac{1}{2}\left(\alpha_1(t) p_1^2 + \alpha_2(t) p_2^2\right) + \alpha_3(t) p_1 p_2 - \frac{1}{2}\left(\beta_1(t) q_1^2 + \beta_2(t) q_2^2\right) - \beta_3(t) q_1 q_2 \,, \tag{12}$$

and constraint equation

$$f_\beta = \dot{q}_2 - b q_1 = 0 \,, \tag{13}$$

where the b is a constant. The generalized force is potential, the equations of motion can be written in the form [33]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = \delta_\beta a_{\varepsilon+\beta,s} \quad (s = 1, 2) \tag{14}$$

where δ_β are the multipliyers of the constraint. Using the Hamiltonian and the generalized momenta for this Lagrangian

$$H = p_s\dot{q}_s - L = \frac{1}{2}\left(\alpha_1(t)p_1^2 + \alpha_2(t)p_2^2\right) + \alpha_3(t)p_1p_2 + \frac{1}{2}\left(\beta_1(t)q_1^2 + \beta_2(t)q_2^2\right) + \beta_3(t)q_1q_2,$$

$$p_s = \frac{\partial L}{\partial \dot{q}_s} \tag{15}$$

we transform equations (4) into

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + (\delta_\beta)\, a_{\varepsilon+\beta,s}, \quad (\beta = 1, \cdots, g) \tag{16}$$

where, before integrating the equations (16), δ_β is expressed by the function of $q_s, \dot{q}_s, t$ and (δ_β) is δ_β expressed as a function of q_s, p_s, t. Substituting Eqs.(12), (13) and (15) into (16), leads to

$$\dot{q}_1 = \alpha_1(t)p_1 + \alpha_3(t)p_2, \quad \dot{q}_2 = \alpha_2(t)p_2 + \alpha_3(t)p_1,$$

$$\dot{p}_1 = -\beta_1(t)q_1 - \beta_3(t)q_2, \quad \dot{p}_2 = -\beta_2(t)q_2 - \beta_3(t)q_1 + (\delta). \tag{17}$$

From Eqs. (13) and (17), we obtain (δ) as

$$(\delta) = \frac{1}{\alpha_2}\left[(k\alpha_1 - \dot{\alpha}_3)p_1 + (k\alpha_3 - \dot{\alpha}_2)p_2 + (\alpha_2\beta_3 + \alpha_3\beta_1)q_1 + (\alpha_3\beta_3 + \alpha_2\beta_2)q_2\right]. \tag{18}$$

In order to express H, as given by (15), in the form (7), we make the following identifications for the phase-space functions Γ_i and the coefficient h_i:

$$\Gamma_1 = \frac{1}{2}p_1^2\,, \quad \Gamma_2 = \frac{1}{2}p_2^2\,, \quad \Gamma_3 = p_1p_2, \quad \Gamma_4 = \frac{1}{2}q_1^2\,, \quad \Gamma_5 = \frac{1}{2}q_2^2\,, \quad \Gamma_6 = q_1q_2\,, \tag{19}$$

$$h_1 = \alpha_1(t)\,, \quad h_2 = \alpha_2(t)\,, \quad h_3 = \alpha_3(t)\,, \quad h_4 = \beta_1(t)\,, \quad h_5 = \beta_2(t)\,, \quad h_6 = \beta_3(t)\,. \tag{20}$$

In this case, the closure of the dynamical algebra requires four more Γ_l, namely

$$\Gamma_7 = p_1q_1\,, \quad \Gamma_8 = p_1q_2\,, \quad \Gamma_9 = p_2q_2\,, \quad \Gamma_{10} = p_2q_1\,,$$

with the corresponding null coefficients $h_7 = h_8 = h_9 = h_{10} = 0$. The nonvanishing Poisson brackets in (7), now turn out to be

$$[\Gamma_1\,,\,\Gamma_4] = -\Gamma_7\,, \quad [\Gamma_1\,,\,\Gamma_6] = -\Gamma_8\,, \quad [\Gamma_1\,,\,\Gamma_7] = -2\Gamma_1\,, \quad [\Gamma_1\,,\,\Gamma_{10}] = -\Gamma_3\,,$$

$$[\Gamma_2\,,\,\Gamma_5] = -\Gamma_9\,, \quad [\Gamma_2\,,\,\Gamma_6] = -\Gamma_{10}\,, \quad [\Gamma_2\,,\,\Gamma_8] = -\Gamma_3\,, \quad [\Gamma_2\,,\,\Gamma_9] = -2\Gamma_2\,,$$

$$[\Gamma_3\,,\,\Gamma_4] = -\Gamma_{10}\,, \quad [\Gamma_3\,,\,\Gamma_5] = -\Gamma_8\,, \quad [\Gamma_3\,,\,\Gamma_6] = -\Gamma_7 - \Gamma_9\,, \quad [\Gamma_3\,,\,\Gamma_7] = -\Gamma_3\,,$$

$$[\Gamma_3\,,\,\Gamma_8] = -2\Gamma_1\,, \quad [\Gamma_3\,,\,\Gamma_9] = -\Gamma_3\,, \quad [\Gamma_3\,,\,\Gamma_{10}] = -2\Gamma_2$$

$$[\Gamma_4\,,\,\Gamma_7] = 2\Gamma_4\,, \quad [\Gamma_4\,,\,\Gamma_8] = \Gamma_6\,, \quad [\Gamma_5\,,\,\Gamma_9] = 2\Gamma_5\,, \quad [\Gamma_5\,,\,\Gamma_{10}] = \Gamma_6\,,$$

$$[\Gamma_6\,,\,\Gamma_7] = \Gamma_6\,, \quad [\Gamma_6\,,\,\Gamma_8] = 2\Gamma_5\,, \quad [\Gamma_6\,,\,\Gamma_9] = \Gamma_6\,, \quad [\Gamma_6\,,\,\Gamma_{10}] = 2\Gamma_4\,,$$

$$[\Gamma_7\,,\,\Gamma_8] = \Gamma_8\,, \quad [\Gamma_7\,,\,\Gamma_{10}] = -\Gamma_{10}\,,$$

$$[\Gamma_8\,,\,\Gamma_9] = \Gamma_8\,, \quad [\Gamma_8\,,\,\Gamma_{10}] = \Gamma_7 - \Gamma_9\,, \quad [\Gamma_9\,,\,\Gamma_{10}] = \Gamma_{10}. \tag{21}$$

Substituting these results in (9) and considering the terms for each Γ_i, yields the following set of differential equations for the λ_s:

$$\dot{\lambda}_1 = -2\alpha_1\lambda_7 - 2\alpha_3\lambda_8 + \frac{2}{\alpha_2}(k\alpha_1 - \dot{\alpha}_3)\lambda_3 \tag{22}$$

$$\dot{\lambda}_2 = -2\alpha_2\lambda_9 - 2\alpha_3\lambda_{10} + \frac{2}{\alpha_2}(k\alpha_3 - \dot{\alpha}_2)\lambda_2 \tag{23}$$

$$\dot{\lambda}_3 = -\alpha_3\lambda_7 - \alpha_2\lambda_8 - \alpha_3\lambda_9 - \alpha_1\lambda_{10} + \frac{1}{\alpha_2}(k\alpha_1 - \dot{\alpha}_3)\lambda_2 + \frac{1}{\alpha_2}(k\alpha_3 - \dot{\alpha}_2)\lambda_3 \tag{24}$$

$$\dot{\lambda}_4 = 2\beta_1\lambda_7 + 2\beta_3\lambda_{10} + \frac{2}{\alpha_2}(\alpha_2\beta_3 + \alpha_3\beta_1)\lambda_{10} \tag{25}$$

$$\dot{\lambda}_5 = 2\beta_3\lambda_8 + 2\beta_2\lambda_9 + \frac{2}{\alpha_2}(\alpha_3\beta_3 + \alpha_2\beta_2)\lambda_9 \tag{26}$$

$$\dot{\lambda}_6 = \beta_3\lambda_7 + \beta_1\lambda_8 + \beta_3\lambda_9 + \beta_2\lambda_{10} + \frac{1}{\alpha_2}(\alpha_2\beta_3 + \alpha_3\beta_1)\lambda_9 + \frac{1}{\alpha_2}(\alpha_3\beta_3 + \alpha_2\beta_2)\lambda_{10} \tag{27}$$

$$\dot{\lambda}_7 = \beta_1\lambda_1 + \beta_3\lambda_3 - \alpha_1\lambda_4 - \alpha_3\lambda_6 + \frac{1}{\alpha_2}(\alpha_2\beta_3 + \alpha_3\beta_1)\lambda_3 + \frac{1}{\alpha_2}(k\alpha_1 - \dot{\alpha}_3)\lambda_{10} \tag{28}$$

$$\dot{\lambda}_8 = \beta_3\lambda_1 + \beta_2\lambda_3 - \alpha_3\lambda_5 - \alpha_1\lambda_6 + \frac{1}{\alpha_2}(\alpha_3\beta_3 + \alpha_2\beta_2)\lambda_3 + \frac{1}{\alpha_2}(k\alpha_1 - \dot{\alpha}_3)\lambda_9 \tag{29}$$

$$\dot{\lambda}_9 = \beta_2\lambda_2 + \beta_3\lambda_3 - \alpha_3\lambda_6 - \alpha_2\lambda_5 + \frac{1}{\alpha_2}(\alpha_3\beta_3 + \alpha_2\beta_2)\lambda_2 + \frac{1}{\alpha_2}(k\alpha_3 - \dot{\alpha}_2)\lambda_9 \tag{30}$$

$$\dot{\lambda}_{10} = \beta_3\lambda_2 + \beta_1\lambda_3 - \alpha_3\lambda_4 - \alpha_2\lambda_6 + \frac{1}{\alpha_2}(\alpha_2\beta_3 + \alpha_3\beta_1)\lambda_2 + \frac{1}{\alpha_2}(k\alpha_3 - \dot{\alpha}_2)\lambda_{10} \tag{31}$$

Since to obtain the general solution of these ten coupled differential equations is a difficult task, we resort to particular solutions of these differential equations and demonstrate the computation of the invariant in the case of equal mass and equal frequency, namely, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$. We also make the following ansatz:

$$\dot{\lambda}_1 = \dot{\lambda}_2 = -2\dot{\psi}(t)\,, \tag{32}$$

leading to

$$\lambda_1 = -2\psi + c_1, \lambda_2 = -2\psi + c_2\,. \tag{33}$$

With all this, equations (22)-(26) take the form

$$\alpha(\lambda_7 - \lambda_9) + \alpha_3(\lambda_8 - \lambda_{10}) = \frac{1}{\alpha}(k\alpha - \dot{\alpha}_3)\lambda_3 - \frac{1}{\alpha}(k\alpha_3 - \dot{\alpha})\lambda_2\,, \tag{34}$$

$$\lambda_8 + \lambda_{10} = 2\dot{\psi}\overline{\alpha}_3 + \overline{\alpha}\dot{\lambda}_3 + \left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - \left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right)\right]\lambda_2 + \left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - \left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right)\right]\lambda_3\,, \tag{35}$$

$$\lambda_7 + \lambda_9 = -\overline{\alpha}_3\dot{\lambda}_3 - 2\overline{\alpha}\dot{\psi} + \left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - \left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right)\right]\lambda_2 + \left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - \left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right)\right]\lambda_3\,, \tag{36}$$

$$\begin{aligned} 2\lambda_8 = &-\frac{\alpha}{\alpha_3}\left(\lambda_7 - \lambda_9\right) + 2\overline{\alpha}_3\dot{\psi} + \overline{\alpha}\dot{\lambda}_3 + \left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - \left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - \frac{1}{\alpha\alpha_3}\left(k\alpha_3 - \dot{\alpha}\right)\right]\lambda_2 \\ &+ \left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - \left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) + \frac{1}{\alpha\alpha_3}\left(k\alpha - \dot{\alpha}_3\right)\right]\lambda_3\,, \end{aligned} \tag{37}$$

$$\begin{aligned} 2\lambda_{10} = &\frac{\alpha}{\alpha_3}\left(\lambda_7 - \lambda_9\right) + 2\overline{\alpha}_3\dot{\psi} + \overline{\alpha}\dot{\lambda}_3 + \left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - \left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) + \frac{1}{\alpha\alpha_3}\left(k\alpha_3 - \dot{\alpha}\right)\right]\lambda_2 \\ &+ \left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - \left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - \frac{1}{\alpha\alpha_3}\left(k\alpha - \dot{\alpha}_3\right)\right]\lambda_3\,, \end{aligned} \tag{38}$$

$$\begin{aligned} \dot{\lambda}_4 = &\left(2\beta + \frac{2\alpha}{\alpha_3}\beta_3\right)\left(\lambda_7 - \lambda_9\right) - 2\beta\overline{\alpha}\dot{\psi} + 2\overline{\alpha}_3\left(2\beta_3 + \frac{\alpha_3}{\alpha}\beta\right)\dot{\psi} + 2\beta_3\overline{\alpha}\dot{\lambda}_3 \\ &+ \left[\left(\frac{2\alpha_3}{\alpha}\beta_3 + \frac{\alpha_3^2}{\alpha^2}\beta - \beta\right)\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - 2\beta_3\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) + \frac{1}{\alpha\alpha_3}\left(2\beta_3 + \frac{\alpha_3}{\alpha}\beta\right)\left(\left(k\alpha_3 - \dot{\alpha}\right)\right)\right]\lambda_2 \\ &+ \left[-2\beta_3\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) + \left(\frac{2\alpha_3}{\alpha}\beta_3 + \frac{\alpha_3^2}{\alpha^2}\beta - \beta\right)\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - \left(\frac{2\beta_3}{\alpha\alpha_3} + \frac{\beta}{\alpha^2}\right)\left(k\alpha - \dot{\alpha}_3\right)\right]\lambda_3\,, \end{aligned} \tag{39}$$

$$\dot{\lambda}_5 = -\left(2\beta + \frac{\alpha}{\alpha_3}\beta + \frac{\alpha_3}{\alpha}\beta_3\right)(\lambda_7 - \lambda_9) + 2\left(\beta_3\overline{\alpha}_3 - \beta\overline{\alpha} - \frac{\overline{\alpha}}{\alpha}\right)\dot{\psi} + \left(\beta_3\overline{\alpha} - 2\beta\overline{\alpha}_3 - \frac{\alpha_3\beta_3}{\alpha}\overline{\alpha}_3\right)\dot{\lambda}_3$$
$$+ \left[\left(\frac{\alpha_3^2}{\alpha^2}\beta_3 + 2\frac{\alpha_3}{\alpha}\beta - \beta_3\right)\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - 2\beta\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - \frac{\beta_3}{\alpha\alpha_3}\left(k\alpha_3 - \dot{\alpha}\right)\right]\lambda_2$$
$$+ \left[\left(\frac{\alpha_3^2}{\alpha^2}\beta_3 + 2\frac{\alpha_3}{\alpha}\beta - \beta_3\right)\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - 2\beta\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) + \frac{\beta}{\alpha\alpha_3}\left(k\alpha - \dot{\alpha}_3\right)\right]\lambda_3\,, \quad (40)$$

where $\overline{\alpha} = \alpha/\left(\alpha^2 - \alpha_3^2\right)$, $\overline{\alpha}_3 = \alpha/\left(\alpha^2 - \alpha_3^2\right)$, $\dot{\overline{\alpha}} = \dot{\alpha}/\left(\alpha^2 - \alpha_3^2\right)$, $\dot{\overline{\alpha}}_3 = \dot{\alpha}_3/\left(\alpha^2 - \alpha_3^2\right)$. An inspection of these results allows us to assume $\dot{\lambda}_3 = 0$ (i.e., $\lambda_3 = c_3$, a constant) and $\lambda_7 = \lambda_9$ for simplicity. This implies quite a simplification of the remaining $\dot{\lambda}_s$ from equations(27)-(31). In particular, one immediately obtains

$$\dot{\lambda}_4 = 2\left[\overline{\alpha}_3\left(2\beta_3 + \frac{\alpha_3}{\alpha}\beta\right) - \beta\overline{\alpha}\right]\dot{\psi}$$
$$+ \left[\left(\frac{2\alpha_3}{\alpha}\beta_3 + \frac{\alpha_3^2}{\alpha^2}\beta - \beta\right)\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - 2\beta_3\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) + \frac{1}{\alpha\alpha_3}\left(2\beta_3 + \frac{\alpha_3}{\alpha}\beta\right)\left(\left(k\alpha_3 - \dot{\alpha}\right)\right)\right]\left(-2\psi + c_2\right)$$
$$+ \left[-2\beta_3\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) + \left(\frac{2\alpha_3}{\alpha}\beta_3 + \frac{\alpha_3^2}{\alpha^2}\beta - \beta\right)\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - \left(\frac{2\beta_3}{\alpha\alpha_3} + \frac{\beta}{\alpha^2}\right)\left(k\alpha - \dot{\alpha}_3\right)\right]c_3\,, \quad (41)$$

$$\dot{\lambda}_5 = 2\left(\beta_3\overline{\alpha}_3 - \beta\overline{\alpha} - \frac{\overline{\alpha}}{\alpha}\right)\dot{\psi}$$
$$+ \left[\left(\frac{\alpha_3^2}{\alpha^2}\beta_3 + 2\frac{\alpha_3}{\alpha}\beta - \beta_3\right)\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - 2\beta\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - \frac{\beta_3}{\alpha\alpha_3}\left(k\alpha_3 - \dot{\alpha}\right)\right]\left(-2\psi + c_2\right)$$
$$+ \left[\left(\frac{\alpha_3^2}{\alpha^2}\beta_3 + 2\frac{\alpha_3}{\alpha}\beta - \beta_3\right)\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - 2\beta\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) + \frac{\beta}{\alpha\alpha_3}\left(k\alpha - \dot{\alpha}_3\right)\right]c_3\,, \quad (42)$$

leading to

$$\lambda_4 = \sigma_1(t) + c_4\,, \quad \lambda_5 = \sigma_2(t) + c_5\,, \quad \lambda_6 = \sigma_3(t) + c_6\,, \quad \lambda_7 = \lambda_9 = \sigma_4\,, \quad \lambda_8 = \sigma_5\,, \quad \lambda_{10} = \sigma_6\,, \quad (43)$$

where

$$\sigma_1(t) = 2\int\left[\overline{\alpha}_3\left(2\beta_3 + \frac{\alpha_3}{\alpha}\beta\right) - \beta\overline{\alpha}\right]\dot{\psi}\,dt +$$
$$\int\left[\left(\frac{2\alpha_3}{\alpha}\beta_3 + \frac{\alpha_3^2}{\alpha^2}\beta - \beta\right)\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - 2\beta_3\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) + \frac{1}{\alpha\alpha_3}\left(2\beta_3 + \frac{\alpha_3}{\alpha}\beta\right)\left(\left(k\alpha_3 - \dot{\alpha}\right)\right)\right]\left(-2\psi + c_2\right)dt$$
$$+ \int\left[-2\beta_3\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) + \left(\frac{2\alpha_3}{\alpha}\beta_3 + \frac{\alpha_3^2}{\alpha^2}\beta - \beta\right)\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - \left(\frac{2\beta_3}{\alpha\alpha_3} + \frac{\beta}{\alpha^2}\right)\left(k\alpha - \dot{\alpha}_3\right)\right]c_3\,dt\,, \quad (44)$$

$$\sigma_2(t) = 2\int\left(\beta_3\overline{\alpha}_3 - \beta\overline{\alpha} - \frac{\overline{\alpha}}{\alpha}\right)\dot{\psi}\,dt +$$
$$\int\left[\left(\frac{\alpha_3^2}{\alpha^2}\beta_3 + 2\frac{\alpha_3}{\alpha}\beta - \beta_3\right)\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) - 2\beta\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - \frac{\beta_3}{\alpha\alpha_3}\left(k\alpha_3 - \dot{\alpha}\right)\right]\left(-2\psi + c_2\right)dt$$
$$+ \int\left[\left(\frac{\alpha_3^2}{\alpha^2}\beta_3 + 2\frac{\alpha_3}{\alpha}\beta - \beta_3\right)\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right) - 2\beta\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) + \frac{\beta}{\alpha\alpha_3}\left(k\alpha - \dot{\alpha}_3\right)\right]c_3\,dt\,, \quad (45)$$

$$\sigma_3 = \int\left[\left(3\beta + \frac{\alpha_3}{\alpha}\beta_3\right)\overline{\alpha}_3 - \left(3\beta_3 + \frac{\alpha_3}{\alpha}\beta\right)\overline{\alpha}\right]\dot{\psi}\,dt +$$
$$\int\left[\left(\alpha_3\beta_3 + \frac{\alpha_3^2}{2\alpha^2}\beta - \frac{3}{2}\beta\right)\left(k\overline{\alpha} - \dot{\overline{\alpha}}_3\right) + \left(\frac{\alpha_3^2}{2\alpha^2}\beta_3 + \frac{\alpha_3}{\alpha}\beta - \frac{3}{2}\beta_3\right)\left(k\overline{\alpha}_3 - \dot{\overline{\alpha}}\right)\right.$$
$$\left. + \frac{1}{\alpha\alpha_3}\left(\frac{\alpha_3\beta_3}{2\alpha} + \frac{\beta}{2}\right)\left(k\alpha - \dot{\alpha}_3\right)\right]\left(-2\psi + c_2\right)\,,dt$$

$$+\int\left[\left(\frac{\alpha_3^2}{2\alpha^2}\beta_3-\frac{3}{2}\beta_3+\frac{\alpha_3}{\alpha}\beta\right)\left(k\overline{\alpha}-\dot{\overline{\alpha}}_3\right)+\right.$$

$$\left.\left(\frac{\alpha_3^2}{2\alpha^2}\beta+\alpha_3\beta_3-\frac{3}{2}\beta\right)\left(k\overline{\alpha}_3-\dot{\overline{\alpha}}\right)-\frac{1}{\alpha\alpha_3}\left(\frac{\alpha_3\beta_3}{2\alpha}+\frac{\beta}{2}\right)\left(k\alpha-\dot{\alpha}_3\right)\right]c_3\,dt\,, \tag{46}$$

$$\sigma_4=-\overline{\alpha}\dot{\psi}+\frac{1}{2}\left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}-\dot{\overline{\alpha}}_3\right)-\left(k\overline{\alpha}_3-\dot{\overline{\alpha}}\right)\right](-2\psi+c_2)+\frac{1}{2}\left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}_3-\dot{\overline{\alpha}}\right)-\left(k\overline{\alpha}-\dot{\overline{\alpha}}_3\right)\right]c_3\,, \tag{47}$$

$$\sigma_5=\overline{\alpha}_3\dot{\psi}+\frac{1}{2}\left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}_3-\dot{\overline{\alpha}}\right)-\left(k\overline{\alpha}-\dot{\overline{\alpha}}_3\right)-\frac{1}{\alpha\alpha_3}\left(k\alpha_3-\dot{\alpha}\right)\right](-2\psi+c_2)$$
$$+\frac{1}{2}\left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}-\dot{\overline{\alpha}}_3\right)-\left(k\overline{\alpha}_3-\dot{\overline{\alpha}}\right)+\frac{1}{\alpha\alpha_3}\left(k\alpha-\dot{\alpha}_3\right)\right]c_3\,, \tag{48}$$

$$\sigma_6=\overline{\alpha}_3\dot{\psi}+\frac{1}{2}\left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}_3-\dot{\overline{\alpha}}\right)-\left(k\overline{\alpha}-\dot{\overline{\alpha}}_3\right)+\frac{1}{\alpha\alpha_3}\left(k\alpha_3-\dot{\alpha}\right)\right](-2\psi+c_2)$$
$$+\frac{1}{2}\left[\frac{\alpha_3}{\alpha}\left(k\overline{\alpha}-\dot{\overline{\alpha}}_3\right)-\left(k\overline{\alpha}_3-\dot{\overline{\alpha}}\right)-\frac{1}{\alpha\alpha_3}\left(k\alpha-\dot{\alpha}_3\right)\right]c_3\,, \tag{49}$$

and $c_i(i=1,2,\cdots,6)$ are constants of integration. In this way all ten λ_i are determined. Furthermore, we set $c_1=c_2=c_4=c_5=c_6=0$, and use these results for λ_i in (8) to obtain the final form of the invariant I, namely,

$$I=-\psi(t)(p_1^2+p_2^2)+c_3p_1p_2+\frac{1}{2}\sigma_1(t)q_1^2+\sigma_2(t)q_2^2+\sigma_3(t)q_1q_2+\sigma_4(t)(p_1q_1+p_2q_2)+\sigma_5(t)p_1q_2+\sigma_6(t)p_2q_1\,, \tag{50}$$

for the system represented by (12) and (13).

3 Conclusions

The present work is a generalization of [19] to linear nonholonomic systems in two dimensions. The exact dynamical invariants are built directly for this dynamical systems. From the example, it is clear that the general construction of exact dynamical invariants for nonholonomic dynamical systems using Lie algebraic method is a difficult task.

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