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Fractionally coupled solutions of the diffusion equation

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Abstract

We analyze a family of solutions of the diffusion equation, which also satisfy an additional fractional equation, in $(1+1)$ and $(3+1)$ dimensions. The solutions may be interpreted either as coupled solutions of the diffusion equation or as a diffusion process with internal degrees of freedom.

Keywords: Fractional derivative; Fourier analysis; Coupled diffusion processes

1. Introduction

Diffusion equations have found many applications in fields as diverse as physics, biology, ecology and the social sciences. A diffusion equation comes into play whenever the time variation of some variable in a space neighborhood is controlled by the net flux (in minus out) of that variable. In its original form

$$\partial_t u(x,t) - \partial_x^2 u(x,t) = 0 \tag{1}$$

the equation takes into account only local effects, the operator kernels associated to the first and second terms being $\delta'(t-t')$ and $\delta''(x-x')$. Non-local

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effects in time and space may be described by non-local kernels or, as already proposed by several authors [1–4] by replacing the first and second derivatives by fractional derivatives. Here we argue that some non-local effects may be taken into account even for solutions of the original diffusion equation if, in addition to Eq. (1), we require the solutions to obey

$$\left(A\partial_t^{\alpha} + B\partial_x^{\beta}\right)u(x,t) = 0 \tag{2}$$

where, with $\alpha, \beta \in R^+, \partial_t^{\alpha}$ and ∂_x^{β} are fractional derivatives and, for consistency, for some power n of the operator

$$\left(A\partial_t^{\alpha} + B\partial_x^{\beta}\right)^n = \partial_t - \partial_x^2 \tag{3}$$

In the simplest case, $\alpha = 1/2$, $\beta = 1$ and $n = 2$, the constraint

$$(A\partial_t^{1/2} + B\partial_x)u(x,t) = 0 \tag{4}$$

together with Eq. (3) means that, in addition to the diffusion equation, the local flow of $u(x,t)$ has a non-local relation to the time evolution. In this particular case, the constraints (3) and (4) require A and B to be, at least, 2×2 matrices. This leads, as an interesting consequence, to a simple mechanism to introduce couplings between different diffusion processes.

Solutions of the diffusion equation may be positive or negative, real or complex quantities. It all depends on the initial conditions. In the case of complex solutions, the real and imaginary parts do not mix and therefore the solution represents two independent diffusion processes. Whenever the coefficients A and B in Eq. (4) are matrices or complex quantities the resulting diffusions become coupled processes. Coupled diffusion processes, as used for example in population dynamics of several species [5], are usually obtained by the introduction of explicit decay and growth interaction factors in the equations. Here a different type of coupling is obtained which, without modifying the diffusion equation, forces the solutions to satisfy an additional constraint.

In Sections 2 and 3 we treat in detail the case $\alpha = 1/2, \beta = 1, n = 2$ for $1+1$ and $1+3$ dimensions. The $1+1$ case was already briefly discussed in a previous work [6].

2. The square root equation in (1+1) dimensions

Taking into account the fact that a solution of the diffusion equation can be written as

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(k)\mathrm{e}^{-k^2 t}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{5}$$

space and time fractional derivatives may be defined in a symmetric way in the framework of the standard Fourier transform

$$\frac{\partial^\alpha u(s)}{\partial s^\alpha} \to (-\mathrm{i}\kappa)^\alpha \hat{u}(\kappa) \tag{6}$$

For the diffusion equation in one space dimension, $\partial_t u(x,t) - \partial_x^2 u(x,t) = 0$, we have four possible consistent definitions of the fractional time derivative associated to the square root operator. The possible definitions are:

$$\frac{\partial^{1/2} u(x,t)}{\partial t^{1/2}} = \int_{-\infty}^{\infty} (\mathrm{i}k)\hat{u}(k)\mathrm{e}^{-k^2 t}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{7}$$

$$\frac{\partial^{1/2} u(x,t)}{\partial t^{1/2}} = \int_{-\infty}^{\infty} (-\mathrm{i}k)\hat{u}(k)\mathrm{e}^{-k^2 t}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{8}$$

$$\frac{\partial^{1/2} u(x,t)}{\partial t^{1/2}} = \int_{-\infty}^{\infty} (\mathrm{i}|k|)\hat{u}(k)\mathrm{e}^{-k^2 t}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{9}$$

$$\frac{\partial^{1/2} u(x,t)}{\partial t^{1/2}} = \int_{-\infty}^{\infty} (-\mathrm{i}|k|)\hat{u}(k)\mathrm{e}^{-k^2 t}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{10}$$

than can be summarized as follows

$$\frac{\partial^{1/2} u(x,t)}{\partial t^{1/2}} = \int_{-\infty}^{\infty} \epsilon\mathrm{i}(\delta k + (1-\delta)|k|)\hat{u}(k)\mathrm{e}^{-k^2 t}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \tag{11}$$

where $\delta = 1, 0$ and $\epsilon = 1, -1$.

The square root equation in one space dimension is:

$$\left(A\frac{\partial^{1/2}}{\partial t^{1/2}} + B\frac{\partial}{\partial x}\right)\psi(x,t) = 0 \tag{12}$$

where Eq. (3) requires A and B to be matrices satisfying the conditions:

$$A^2 = I, \quad B^2 = -I \tag{13}$$

$$\{A, B\} \equiv AB + BA = 0 \tag{14}$$

being $\psi(x,t)$ multidimensional with at least two scalar space-time components. Each scalar component satisfies the diffusion equation. The components may be interpreted either as complex diffusion solutions associated to internal degrees of freedom or as coupled diffusion processes. In [6] the solutions $\psi(x,t)$ were named *diffunors.*

Let us consider the following realization of the above algebra in terms of real matrices 2×2 associated to the Pauli matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{15}$$

From Eq. (5) it follows that the solution of the fractional equation (12) may be written as follows

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \begin{pmatrix} \int_{-\infty}^{\infty} \hat{u}_1(k) e^{-k^2 t} e^{-ikx} \, dk \\ \int_{-\infty}^{\infty} \hat{u}_2(k) e^{-k^2 t} e^{-ikx} \, dk \end{pmatrix} \tag{16}$$

By inserting it into (12) we obtain

$$\begin{pmatrix} \epsilon(\delta k + (1-\delta)|k|) & -k \\ k & -\epsilon(\delta k + (1-\delta)|k|) \end{pmatrix} \begin{pmatrix} \hat{u}_1(k) \\ \hat{u}_2(k) \end{pmatrix} = 0. \tag{17}$$

according to the different possible definitions of the time fractional derivative. In this context, the solutions are the following

$$\begin{pmatrix} \hat{u}_1(k) \\ \hat{u}_2(k) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \begin{pmatrix} 1 \\ \frac{k}{|k|} \end{pmatrix}; \begin{pmatrix} 1 \\ -\frac{k}{|k|} \end{pmatrix} \times \phi(k) \tag{18}$$

associated respectively to the possible values of the parameters: $\delta = 0$, $\epsilon = 1$; $\delta = 0$, $\epsilon = -1$; $\delta = 1$, $\epsilon = 1$; $\delta = 1$, $\epsilon = -1$, and being $\phi(k)$ an arbitrary function of k. From the above expression, we obtain that each one of the two components in Eq. (12) is complex and, thus, it might be associated to two real diffusion processes. For each choice in Eq. (18) one obtains therefore four coupled real solutions of the scalar diffusion equation.

As an example, let us consider the case $\phi(k) = e^{-\lambda k^2}$ and the choice of the fractional derivative parameter $\delta = 0$. We get the following initial conditions

$$\begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \begin{pmatrix} \left(\frac{\pi}{\lambda}\right)^{1/2} e^{-x^2/4\lambda} \\ -2\epsilon i \frac{x}{2\lambda} \sum_{k=1}^{\infty} \frac{1}{(2k-1)!!} \left(-\frac{x^2}{2\lambda}\right)^{k-1} \end{pmatrix} \tag{19}$$

which generate the solutions associated to the two possible values of ϵ:

$$\begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = \begin{pmatrix} \left(\frac{\pi}{\lambda+t}\right)^{1/2} e^{-x^2/4(\lambda+t)} \\ -2\epsilon i \frac{x}{2(\lambda+t)} \sum_{k=1}^{\infty} \frac{1}{(2k-1)!!} \left(-\frac{x^2}{2(\lambda+t)}\right)^{k-1} \end{pmatrix} \tag{20}$$

In this case, the first component u_1 is pure real while the second one, u_2, is pure imaginary and it is given in terms of a degenerate hypergeometric function. In this case we have two real coupled processes.

3. The square root equation in (3+1) dimensions

A natural generalization of Eq. (12) to the three space dimensions is the following

$$\left(A\frac{\partial^{1/2}}{\partial t^{1/2}}+\vec{B}.\vec{\nabla}\right)\psi(x,t)=0 \tag{21}$$

where $\psi(x,t)$ is a set of four complex functions, A and B_i are 4×4 Dirac matrices [7] satisfying the conditions:

$$A^2=I,\quad \{B_i,B_j\}=-2\delta_{ij},\quad \{A,B_i\}=0 \tag{22}$$

Taking into account that every component of (21) must satisfy the three dimensional diffusion equation $u_t-\Delta u=0$, being

$$u(\vec{x},t)=\int_{-\infty}^{\infty}\hat{u}(\vec{k})\mathrm{e}^{-\vec{k}^2t}\mathrm{e}^{-\mathrm{i}\vec{k}.\vec{x}}\,\mathrm{d}^3k \tag{23}$$

we can write the solution of the fractional equation (21) as follows

$$\begin{pmatrix}u_1(\vec{x},t)\\u_2(\vec{x},t)\\u_3(\vec{x},t)\\u_4(\vec{x},t)\end{pmatrix}=\begin{pmatrix}\int_{-\infty}^{\infty}\hat{u}_1(\vec{k})\mathrm{e}^{-\vec{k}^2t}\mathrm{e}^{-\mathrm{i}\vec{k}.\vec{x}}\,\mathrm{d}^3k\\\int_{-\infty}^{\infty}\hat{u}_2(\vec{k})\mathrm{e}^{-\vec{k}^2t}\mathrm{e}^{-\mathrm{i}\vec{k}.\vec{x}}\,\mathrm{d}^3k\\\int_{-\infty}^{\infty}\hat{u}_3(\vec{k})\mathrm{e}^{-\vec{k}^2t}\mathrm{e}^{-\mathrm{i}\vec{k}.\vec{x}}\,\mathrm{d}^3k\\\int_{-\infty}^{\infty}\hat{u}_4(\vec{k})\mathrm{e}^{-\vec{k}^2t}\mathrm{e}^{-\mathrm{i}\vec{k}.\vec{x}}\,\mathrm{d}^3k\end{pmatrix} \tag{24}$$

with $\vec{k}=(k_1,k_2,k_3)$. Inserting into (21) for the case $\delta=0$, $\epsilon=1$, we obtain:

$$\begin{pmatrix}|\vec{k}| & 0 & -k_3 & -k_1+\mathrm{i}k_2\\0 & |\vec{k}| & -k_1-\mathrm{i}k_2 & k_3\\k_3 & k_1-\mathrm{i}k_2 & -|\vec{k}| & 0\\k_1+\mathrm{i}k_2 & -k_3 & -|\vec{k}| & \end{pmatrix}\begin{pmatrix}\hat{u}_1(\vec{k})\\\hat{u}_2(\vec{k})\\\hat{u}_3(\vec{k})\\\hat{u}_4(\vec{k})\end{pmatrix}=0. \tag{25}$$

which gives a set of four basic solutions in the Fourier space:

$$\begin{pmatrix}\hat{u}_1(\vec{k})\\\hat{u}_2(\vec{k})\\\hat{u}_3(\vec{k})\\\hat{u}_4(\vec{k})\end{pmatrix}=\begin{pmatrix}1\\0\\\frac{k_3}{|\vec{k}|}\\\frac{k_1+\mathrm{i}k_2}{|\vec{k}|}\end{pmatrix};\begin{pmatrix}0\\1\\\frac{k_1-\mathrm{i}k_2}{|\vec{k}|}\\-\frac{k_3}{|\vec{k}|}\end{pmatrix};\begin{pmatrix}\frac{k_3}{|\vec{k}|}\\\frac{k_1+\mathrm{i}k_2}{|\vec{k}|}\\1\\0\end{pmatrix};\begin{pmatrix}\frac{k_1-\mathrm{i}k_2}{|\vec{k}|}\\-\frac{k_3}{|\vec{k}|}\\0\\1\end{pmatrix}\times\phi(\vec{k}) \tag{26}$$

Thus, each solution of (21) is associated to four complex diffusion processes or, equivalently, to eight real diffusion processes, with initial conditions related by the above Fourier transforms. Similar results are obtained for the other values of the parameters δ and ϵ.

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