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A Numerical Study of Fractional Evolution-Diffusion Dirac-like Equations

T. Pierantozzi† and L. Vázquez†‡
† Department of Applied Mathematics
Faculty of Informatics, Universidad Complutense of Madrid, Spain
‡ Centre for Astrobiology (CSIC-INTA)
Torrejón de Ardoz, Madrid, Spain

Abstract

Through the Fractional Calculus and following the method used by Dirac to obtain his well-known equation from the Klein-Gordon one, a possible interpolation between the Dirac, diffusion and wave equations in one space dimension can be derived, that we named fractional evolution-diffusion equation Dirac like. Such an equation contains a fractional derivative of order α varying in $(0, 1]$ in time and a first order partial derivative in space. It can be seen as one of the two roots that we would obtain operating a kind of square root of the time fractional diffusion equation in one space dimension, with fractional derivative in time of order $\alpha \in [1, 2]$. Solutions of this equation could model the diffusion of particles whose behavior depends not only on the space and time coordinates, but also on their internal structures. A numerical scheme based on convolution quadrature formula is given for solving this equation and the associated stability bounds are checked in some concrete case.

Keywords: fractional differential equations, Mittag-Leffler and Wright functions, Dirac-type equations, finite difference methods, stability analysis.

1 Introduction

The Fractional Calculus (see [1] or [2], for example) represents a natural instrument to model nonlocal phenomena either in space or time that involve different scales. There are several pathways to the use of fractional models in different applied fields; for instance, see [3] and [4]. In particular, in the case when mathematical models are connected with anomalous diffusion, we could use the CTRW approach introduced by [5], or operate through the Langevin equation approach (see [6]) involving the generalized central limit theorem and the Lévy stable distributions, or by generalization of the classical first Fick‘s law combined with the conservation law (for example, see [7]) and others.

In this paper, we present a generalization of the linear one-dimensional diffusion and wave equations obtained by combining the fractional derivatives and the internal degrees of freedom associated to the system.

Actually, taking into account that the free Dirac equation is, in some sense, the square root of the Klein-Gordon equation (see, for instance, [8]), in a similar way we can operate a kind of square root of the time fractional diffusion equation in one space dimension (see [9], [3], [10], [11], [12]) given by

$$\partial_t^{2\alpha} u(t,x) - \partial_{xx} u(t,x) = 0, \tag{1}$$

where $0 < \alpha \leq 1$, through the system of fractional evolution-diffusion equations Dirac like

$$(A\partial_t^\alpha + B\partial_x)\,\psi(t,x) = 0, \quad \psi(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}, \tag{2}$$

with $0 < \alpha \leq 1$ and where A and B are 2×2 matrices satisfying Pauli's algebra, this is:

$$A^2 = I, \quad B^2 = -I, \quad AB + BA = 0,$$

being I the identity operator.

System (2) has been previously introduced in the references [13], [14], [15] and [16]. Each component of $\psi(t,x)$ satisfies (1) while the index property ${\partial_t}^\alpha {\partial_t}^\alpha u = {\partial_t}^{2\alpha} u$ holds. Thus, in the interval $1/2 < \alpha < 1$, the system of fractional evolution equations Dirac like (2) represents a fractional interpolation between the diffusion ($\alpha = 1/2$) and wave ($\alpha = 1$) equations.

Solutions of this system could model the diffusion of particles whose behavior depends not only on the space and time coordinates, but also on their internal structures.

Now, if we shrink the study to the pure real matrices of Pauli's type leading to a system (2) of separated equations, we reduce to the pair:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Substituting (3) into (2), it reduces to the following system of equations

$$\begin{cases} \partial_t^\alpha u_1(t,x) - \partial_x u_1(t,x) = 0 \\ \\ \partial_t^\alpha u_2(t,x) + \partial_x u_2(t,x) = 0 \end{cases}, \tag{4}$$

where $0 < \alpha \leq 1$.

This paper is devoted to the construction of numerical schemes solving each component of the system (4) when it is written as the specific fractional evolution-diffusion equation

$$({}^C_0D_t^\alpha u)(t,x) + \lambda \frac{\partial u(t,x)}{\partial x} = 0, \quad x > a,\, t > 0, \tag{5}$$

and together with initial and boundary conditions

$$u(0+, x) = u_0(x), \ x > a, \tag{6}$$

$$u(t, a+) = r(t), \ t > 0, \tag{7}$$

where $a \in \mathbb{R}$, $u_0(x)$ and $r(t)$ are known functions, ${}^C_0D^\alpha_t$ is the Caputo fractional derivative of order α, with $0 < \alpha \leq 1$, and $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Given the Riemann-Liouville derivative of order $\alpha > 0$ (see [1] for example),

$$({}^{RL}_aD^\alpha_x f)(x) = \frac{d^n}{dx^n} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \tag{8}$$

with $x > a$, $n = -[-\alpha]$, the Caputo fractional derivative can be considered as a regularised version of this fractional differential operator since it takes the form

$$({}^C_aD^\alpha_x f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^n(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau. \tag{9}$$

There exists the following relation between the above definitions

$$({}^C_aD^\alpha_x f)(x) = {}^{RL}_aD^\alpha_x \left[f(x) - \sum_{j=0}^{n-1} f^{(j)}(a+) \frac{(x-a)^j}{j!} \right], \tag{10}$$

and, a condition under which both derivatives hold is that $f \in AC^{n-1}(a, \infty)$. Equivalence (10) allows to use pure initial conditions of the classical type when dealing with fractional equations involving Riemann-Liouville or Caputo derivatives. As well, relation (10) justify to restrict the attention on just one definition of the fractional derivative, among the two ones given above, when we look for a corresponding finite difference formula.

In the literature, numerical formulas approximating the fractional derivative are typically obtained for the Riemann-Liouville one.

In our case, we employ a convolution quadrature formula proposed by K. Diethelm [17] for approximating this time fractional derivative and usual finite difference formulas for the space partial derivative. The stability bounds of this scheme, resulting from a previous discrete von Neumann type analysis, are checked in some representative examples when we know the underlying exact analytical results.

2 Solution of the initial-boundary value problem

In order to solve the initial boundary-value problem given by equation (5) together with initial and boundary conditions (6) and (7), we look for a solution $u(t, x)$ in the space of functions whose Laplace transform in time,

$$(\mathcal{L}_t u)(s,x) = \int\limits_0^\infty e^{-st} u(t,x)\, dt, \tag{11}$$

exists for any fixed $x \in [a, +\infty)$. The inverse Laplace transform with respect to s is given by:

$$(\mathcal{L}_s^{-1} u)(t,x) = \frac{1}{2\pi i} \int\limits_{\gamma+i\infty}^{\gamma-i\infty} e^{st} u(s,x)\, ds, \tag{12}$$

with a fixed $\gamma \in \mathbb{R}$, and the following relation holds:

$$(\mathcal{L}_s^{-1} \mathcal{L}_t u)(t,x) = u(t,x). \tag{13}$$

We use the following formula for the Laplace transform of Caputo derivative (9) [2, (2.140)]

$$(\mathcal{L}_t\, {}^C D_t^\alpha u)(s,x) = s^\alpha (\mathcal{L}_t u)(s,x) - s^{\alpha-1} u(0+,x), \tag{14}$$

where $0 < \alpha \leq 1$, s^α is understood, as usually, as the corresponding value of the main branch of the analytic function s^α in the complex plane s with the cut along the positive half-axis $\mathbb{R}_+$. Note that (14) yields the known formula for the Laplace transform of the usual derivative when $\alpha = 1$. Then, applying the Laplace transform (11) to the equation (5) and taking the condition $u(0+,x) = g(x)$ into account, we have:

$$s^\alpha (\mathcal{L}_t u)(s,x) - s^{\alpha-1} u_0(x) + \lambda \frac{\partial}{\partial x} (\mathcal{L}_t u)(s,x) = 0. \tag{15}$$

Multiplying this expression by $e^{xs^\alpha/\lambda}$ and integrating in x between a and x, it results:

$$\lambda e^{\frac{xs^\alpha}{\lambda}} (\mathcal{L}_t u)(s,x) - \lambda e^{\frac{as^\alpha}{\lambda}} (\mathcal{L}_t\, r)(s) - \int\limits_a^x s^{\alpha-1} e^{\frac{xs^\alpha}{\lambda}} u_0(x)\, dx = 0, \tag{16}$$

being $(\mathcal{L}_t u)(s, a+) = (\mathcal{L}_t\, r)(s)$.

Now, we can obtain the explicit formula for the solution $u(t,x)$ by specifying the form of initial and boundary conditions in (16). Let we realize it for the particular initial condition: $u_0(x) = e^{-\mu x}$, where $\mu > 0$ is a constant.

If we insert this function in (16), then we obtain, when $x > a$:

$$\begin{gathered}(\mathcal{L}_t u)(s,x) = e^{\frac{-(x-a)s^\alpha}{\lambda}} (\mathcal{L}_t\, r)(s) + \frac{e^{-\mu x}}{s} - \\ \frac{e^{\frac{-(x-a)s^\alpha}{\lambda}} e^{-\mu a}}{s} + \frac{\mu\lambda}{s} \frac{e^{-\mu x}}{(s^\alpha - \mu\lambda)} - \frac{\mu\lambda e^{\frac{-(x-a)s^\alpha}{\lambda}} e^{-\mu a}}{s(s^\alpha - \mu\lambda)}\end{gathered}\;. \tag{17}$$

Before applying the inverse Laplace transform (12) to (17), we need to introduce the following special functions ([18, Section 18.1], [2]): the biparametric Mittag-Leffler function $E_{\alpha,\beta}(z)$ defined for $\{z, \alpha, \beta\} \in \mathbb{C}$ by

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}, \tag{18}$$

and the Wright function

$$W(z;\alpha,\beta)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)n!}, \tag{19}$$

defined for $z\in\mathbb{C}$ and parameters $\alpha>-1$, $\beta>0$.

It can be shown [2, Sec.1.2.] that, when $t>0$, the following properties hold :

$$\left(\mathcal{L}_t(t^{\alpha-1}E_{\alpha,\alpha}(\mu\lambda t^\alpha))\right)(s)=\frac{1}{(s^\alpha-\mu\lambda)}, \tag{20}$$

$$\left(\mathcal{L}_t(t^{\alpha}E_{\alpha,\alpha+1}(\mu\lambda t^\alpha))\right)(s)=\frac{1}{s\,(s^\alpha-\mu\lambda)}. \tag{21}$$

As well, it can be proved that

$$\left(\mathcal{L}_t\left(\frac{1}{t}W(-\frac{\gamma}{t^\alpha};-\alpha,0)\right)\right)(s)=e^{-\gamma s^\alpha} \tag{22}$$

when $\gamma\neq 0$, $0<\alpha<1$ and

$$\lim_{\alpha\to 1^-}\left(\mathcal{L}_t\left(\frac{1}{t}W(-\frac{\gamma}{t^\alpha};-\alpha,0)\right)\right)(s)=\left(\mathcal{L}_t\,\delta(t-\gamma)\right)(s),$$

where $\delta(t)$ is the generalized delta function.

So, inverting equation (17) we have, when $x>a$ and $0<\alpha<1$:

$$\begin{aligned}u(t,x)&=\int_0^t r(t-\tau)\tfrac{1}{\tau}W(-\tfrac{(x-a)}{\lambda\tau^\alpha};-\alpha,0)\,d\tau+e^{-\mu x}-e^{-\mu a}\int_0^t\tfrac{1}{\tau}W(-\tfrac{(x-a)}{\lambda\tau^\alpha};-\alpha,0)\,d\tau+\\&+e^{-\mu x}\mu\lambda t^\alpha E_{\alpha,\alpha+1}(\mu\lambda t^\alpha)-\mu\lambda e^{-\mu a}\int_0^t\tfrac{(t-\tau)^\alpha}{\tau}E_{\alpha,\alpha+1}(\mu\lambda(t-\tau)^\alpha)W(-\tfrac{(x-a)}{\lambda\tau^\alpha};-\alpha,0)\,d\tau.\end{aligned} \tag{23}$$

When $\alpha=1$, taking into account that $\mu\lambda tE_{1,2}(\mu\lambda t)=e^{\mu\lambda t}-1$ and completing the solution given by (23) when $\alpha\to 1$ with initial and boundary conditions, it results

$$u(t,x)=\left\{\begin{array}{ll}e^{-\mu(x-\lambda t)} & 0\leq t<\frac{(x-a)}{\lambda}\\ r(t-\frac{(x-a)}{\lambda}) & t\geq\frac{(x-a)}{\lambda}\end{array}\right., \tag{24}$$

for $\lambda>0$ and

$$u(t,x)=\left\{\begin{array}{ll}e^{-\mu(x-\lambda t)} & x>a,\ t\geq 0\\ r(t) & x=a,\ t>0\end{array}\right., \tag{25}$$

for $\lambda<0$. In order to be this solution continuous, it has to turn out that $r(0)=e^{-\mu a}$.

Now, if we observe that

$$\mu\lambda t^\alpha E_{\alpha,\alpha+1}(\mu\lambda t^\alpha)=E_{\alpha,1}(\mu\lambda t^\alpha)-1,$$

we can rewrite solution (23), where $x > a$ and $0 < \alpha < 1$, as follows:

$$u(t,x) = e^{-\mu x} E_{\alpha,1}(\mu\lambda t^{\alpha}) + \int_0^t \left(r(t-\tau) - e^{-\mu a} E_{\alpha,1}(\mu\lambda (t-\tau)^{\alpha})\right) \tfrac{1}{\tau} W(-\tfrac{(x-a)}{\lambda \tau^{\alpha}}; -\alpha, 0)\, d\tau; \tag{26}$$

then, when $r(t) = e^{-\mu a} E_{\alpha,1}(\mu\lambda t^{\alpha})$, the solution is reduced to be:

$$u(t,x) = e^{-\mu x} E_{\alpha,1}(\mu\lambda t^{\alpha}), \tag{27}$$

for all $x \geq a$ and $t \geq 0$.

3 Construction of the numerical schemes

The most important feature of a fractional differential derivative that has to be taken into account when constructing a numerical scheme is its non-local property. This feature leads to discretizations consisting of a lower triangular matrix instead of a multi-diagonal matrix like in the case of classical derivatives of integer order.

In fact, if we suppose that $t \in [0,T]$ and $x \in [a,b]$, and we introduce the equispaced temporal nodes $t_n = n\Delta t$, where $n = 0,...,N$, $t_0 = 0$, $t_N = T$, and the equispaced spatial nodes $x_l = a + l\Delta x$ where $l = 0,...,M$, $x_0 = a$ and $x_M = b$, then the Riemann-Liouville fractional derivative $({}^{RL}{}_0D_t^{\alpha} u)(t,x)$ can be defined by

$$\frac{1}{\Delta t^{\alpha}} \begin{pmatrix} \omega_{0,0} & 0 & \cdots & \cdots\ 0 \\ \omega_{1,0} & \omega_{1,1} & 0 & \cdots\ 0 \\ \vdots & \vdots & \ddots & \ddots\ 0 \\ \omega_{N,0} & \cdots & \cdots & \omega_{N,N} \end{pmatrix} \begin{pmatrix} u(t_0,x) \\ u(t_1,x) \\ \vdots \\ u(t_N,x) \end{pmatrix}.$$

In an equivalent form, it can be written that the value of the Riemann-Liouville derivative for each time-space point (t_n, x_l) can be approximated by

$$({}^{RL}{}_0D_{t_n}^{\alpha} u)(t_n, x_l) \approx \frac{1}{(\Delta t)^{\alpha}} \sum_{j=0}^{n} \omega_{n,j}\ u_l^j, \tag{28}$$

where u_l^n is the numerical approximation of $u(t_n, x_l)$ and it results $u_0^n = u(t_n, a) = r(t_n)$ and $u_l^0 = u(0, x_l) = u_0(x_l)$, being $r(t)$ a given function as well as $u_0(x)$.

Due to (10), the corresponding approximation of the Caputo derivative is

$$({}^{C}{}_0D_{t_n}^{\alpha} u)(t_n, x_l) \approx \frac{1}{\Delta t^{\alpha}} \sum_{j=0}^{n} \omega_{n,j}\ u_l^j - \frac{t_n^{-\alpha} u_0(x_l)}{\Gamma(1-\alpha)}. \tag{29}$$

In this paper we employ the convolution quadrature formula (28) with the weights $\omega_{n,j}$ proposed by K. Diethelm [17]:

$$\Gamma(2-\alpha)\omega_{n,j} = \begin{cases} 1 & j = n \\ (n-j-1)^{1-\alpha} - 2(n-j)^{1-\alpha} + \\ +(n-j+1)^{1-\alpha} & 1 \leq j \leq n-1 \\ (n-1)^{1-\alpha} - (\alpha-1)n^{-\alpha} - n^{1-\alpha} & j = 0 \end{cases}. \tag{30}$$

It has been shown that the convergence order associated to this formula is $O(\Delta t)^{2-\alpha}$.

In combination with the discrete formula (28) with Diethelm's weight for the time fractional derivative, we employed a classical forward Euler formula to approximate the first order space derivative:

$$\frac{\partial u}{\partial x}(t_n, x_l) \approx \frac{u_{l+1}^n - u_l^n}{\Delta x}. \tag{31}$$

Then, if we write the fractional evolution-diffusion equation (5) by means of the following integral equation with a strongly singular kernel

$$\frac{1}{\Gamma(-\alpha)} \int_0^t \frac{u(\tau, x)}{(t-\tau)^{\alpha+1}}\, d\tau - \frac{t^{-\alpha} u_0(x)}{\Gamma(1-\alpha)} + \lambda \frac{\partial u(t, x)}{\partial x} = 0, \tag{32}$$

the corresponding finite difference equation, when we use formula (28) with weights (30) instead of the time fractional derivative and formula (31) for the space derivative, will be:

$$u_{l+1}^n = \left(1 - \omega_{n,n} \tfrac{\Delta x}{\lambda \Delta t^\alpha}\right) u_l^n + \tfrac{\Delta x}{\lambda} \left[\tfrac{u_0(x_l) t_n^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{j=0}^{n-1} \tfrac{\omega_{n,j}}{\Delta t^\alpha}\, u_l^j \right], \tag{33}$$

for all $l = 0, ..., M$ and $n = 1, ..., N$, and the associated error is $O(\Delta t)^{2-\alpha} + O(\Delta x)$.

A discrete von Neumann type analysis of stability of the scheme (33) together with the initial and boundary conditions (35) [19] highlights that a necessary condition for this scheme being stable for all α, where $0 < \alpha < 1$, is:

$$\frac{\Delta x}{\lambda \Delta t^\alpha \Gamma(2-\alpha)} \leq 1. \tag{34}$$

Now, we can show the numerical results emerging from the simulations of the evolution-diffusion equation (5) together with the initial-boundary value problem

$$\begin{aligned} u(0+, x) &= e^{-\mu x},\ \ x > a, \\ u(t, a+) &= e^{-\mu a} E_{\alpha,1}(\mu \lambda t^\alpha),\ \ t > 0, \end{aligned} \tag{35}$$

where $a \in \mathbb{R}$, $\mu > 0$ and $0 < \alpha < 1$, performed by means of the scheme (33), in order to check the stability bounds (34).

The analytical solution (27) of this problem, according with the properties of the Mittag-Leffler function, takes the specific form

$$u_{1/2}(t, x) = e^{-\mu x} e^{\mu^2 \lambda^2 t} \operatorname{erfc}\left(-\mu \lambda \sqrt{t}\right). \tag{36}$$

when $\alpha = 1/2$, and it reduces to

$$u_1(t, x) = e^{-\mu(x - \lambda t)}, \tag{37}$$

when $\alpha = 1$.

In practice, being $E_{\alpha,1}(\mu\lambda t^{\alpha})$ the exact solution of the ordinary differential equation

$$({}_0^C D_t^{\alpha} u)(t) - \mu\lambda u(t) = 0, \tag{38}$$

with the initial condition $u(0) = 1$, the values of this series are obtained solving numerically this problem.

Fig. 1 represents the function $E_{\alpha,1}(\mu\lambda t^{\alpha})$ for $\mu = \lambda = 1$, $t \in [0,3]$ and when $\alpha = 0.2$, $\alpha = 0.5$, $\alpha = 0.8$ and $\alpha = 1$; the behavior of the analytical solution (27) when $x \in [-2,2]$ and $t \in [0,4]$ is shown for $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.99$ in Fig. 2 when $\mu = \lambda = 1$ and in Fig. 3 when $\mu = 1$ and $\lambda = -1$.

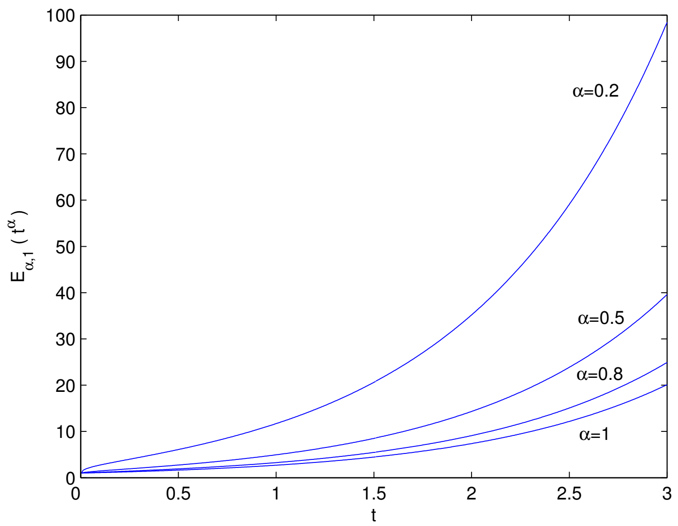

Figure 1: Function $E_{\alpha,1}(t^{\alpha})$ for $0 \leq t \leq 3$ when $\alpha = 0.2$, $\alpha = 0.5$, $\alpha = 0.8$ and $\alpha = 1$.

4 Results and conclusions

To perform the all mentioned numerical simulations, we used the software Matlab7.0 working in double precision. An interesting example confirming that the condition (34) over the time and space steps of the finite scheme (33) is necessary in order to have stability, is the following: if we calculate the maximum of the absolute errors resulting between the values of the exact solution (27), numerically calculated over the space-time grid points, and the approximated ones produced by the implementation of the difference scheme (33) when $\mu = \lambda = 1$, $x \in [1\,,3.5]$, $\Delta x = 0.025$, $t \in [0,2]$, $\Delta t = 0.0125$, we observe that its value is 0.1189 when $\alpha = 0.1$, 0.00241 when $\alpha = 0.5$ and that it is unbounded when $\alpha = 0.9$.

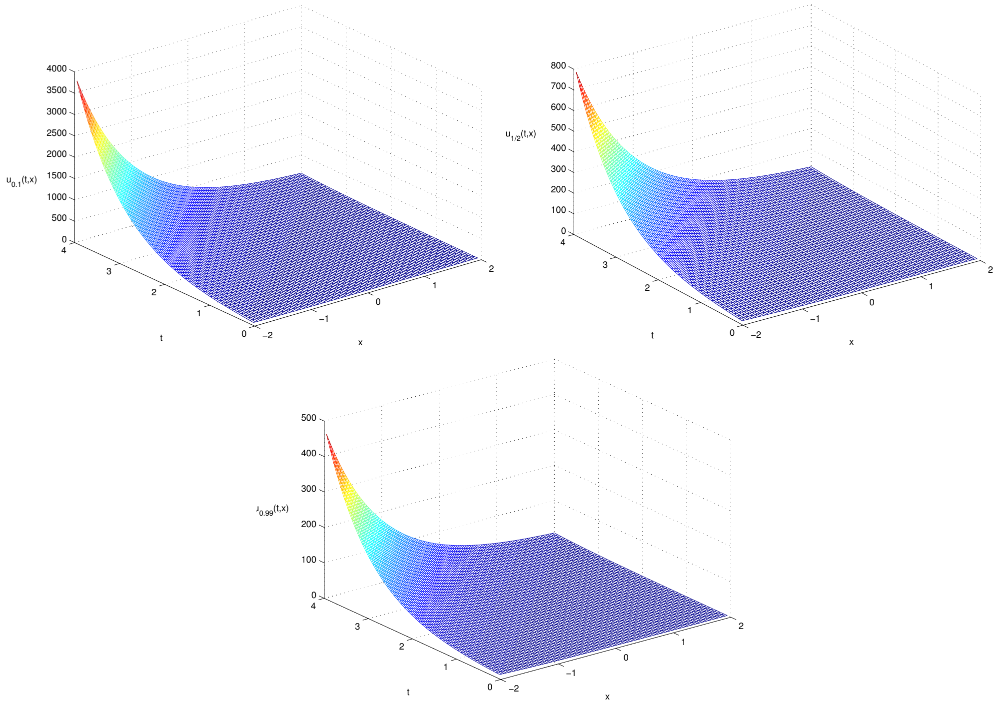

Figure 2: Function $u_\alpha(t,x) = e^{-x} E_{\alpha,1}\left(t^\alpha\right)$ when $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.99$.

The result concerning $\alpha = 0.9$, is due to the breaking of condition (34). In fact, relation $(\Delta x/(\Delta t^\alpha \Gamma(2-\alpha))) \leq 1$ comes true if $\alpha = 0.1$ and $\alpha = 0.5$, whereas it is not fulfilled when $\alpha = 0.9$, being $\Delta x = 0.025$, $\Delta t^{0.9} = 0.0194$ and $\Gamma(1.1) = 0.9513$. Now, if we simulate the same solutions when $\mu = 1$, $x \in [1\,,\ 3.5]$, $\Delta x = 0.025$, $t \in [0, 2]$, $\Delta t = 0.0125$, but $\lambda = -1$, we find that the scheme (33) converges only when $\alpha = 0.1$ and it diverges for $\alpha = 0.5$ and $\alpha = 0.9$.

Table 1 collects the results we obtained concerning the simulations of exact and approximated solution of equation (5) with the conditions (35) when $\mu = 1$, $x \in [1\,,\ 1.1]$, $\Delta x = 0.0001$, $t \in [0, 3]$, $\Delta t = 0.2$, and $\lambda = \pm 1$. Having a Δx of order 10^{-4} facing with a Δt of order 10^{-1} implies the convergence of the scheme (33) for all the values of α such that $0 < \alpha < 1$; in order to have such a small space step we have been obliged to work in a very narrow interval $[a, b]$, due to the limitations of the memory size of the same software we used.

The wide number of simulations we performed for different values of α, Δx, Δt and λ, indicates that a necessary and sufficient condition for the stability of the difference scheme (33) be ensured should be almost stronger than the pure necessary condition we give in (34).

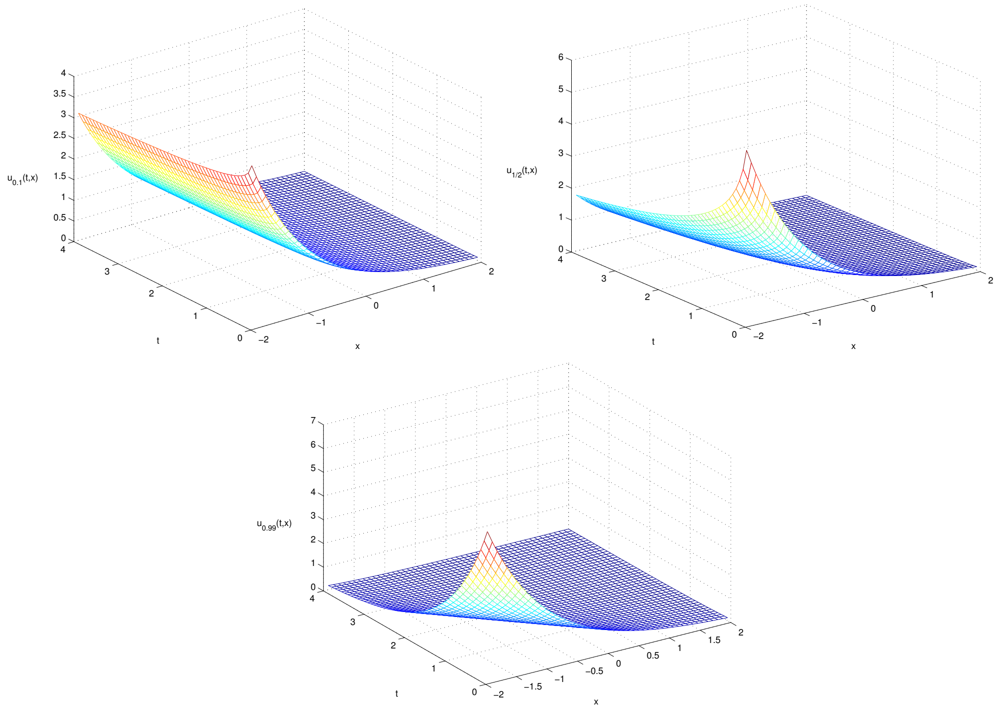

Figure 3: Function $u_\alpha(t,x) = e^{-x} E_{\alpha,1}(-t^\alpha)$ when $\alpha = 0.1$, $\alpha = 0.5$ and $\alpha = 0.99$.

λ	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 0.9$
1	3.20e-04	7.03e-05	4.81e-05
-1	1.02e-06	1.43e-06	1.18e-06

Table 1: Maximum of the absolute errors between the values of the exact solution of (5) with (35) and the approximated ones calculated through the scheme (33) when $\mu = 1$, $x \in [1\,,\,1.1]$, $\Delta x = 0.0001$, $t \in [0,3]$, $\Delta t = 0.2$, for different values of α.

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References

[1] S.G. Samko, A.A. Kilbas and O.I Marichev, *"Fractional Integrals and Derivatives. Theory and Applications"*, Gordon and Breach, New York, 1993.

[2] I. Podlubny, *"Fractional Differential Equations"*, Academic Press, San-Diego, 1999.
[3] R. Metzler and J. Klafter, *"The Random Work's Guide to Anomalous Diffusion: A Fractional dynamics approach"*, Phys. Reports, 339, n.1, 1-77, 2000.
[4] I. Sokolov, J. Klafter and A. Blumen, *"Fractional kinetics"*, Physics Today, 55, 48-54, 2002.
[5] E.W. Montroll, *"Random walks in multidimensional spaces, especially on periodic lattices"*, J. SIAM,4, n.4, 241-260, 1956.
[6] H.C. Fogedby, *"Langevin-equations for continuous-time Levy flights"*, Phys. Rev. E, 50, n.2, 1657-1667, 1994.
[7] P. Paradisi, R. Cesari and F. Mainardi, *"The fractional Fick's law for non-local transport processes"*, Physica A, 293, 130-142, 2001.
[8] S.S. Schweber, *"An introduction to Relativistic Quantum Field Theory"*, Harper Int., New York, 1966.
[9] W.R. Schneider and W. Wyss, *"Fractional diffusion and wave equations"*, J. Math. Phys., 30, n.1, 134-144, 1989.
[10] R. Metzler and T.F. Nonnemacher, *"Space- and Time-fractional diffusion and wave equations, fractional Fokker-Planck equations, and physical motivation"*, Chem. Phys., 284, n.1, 67-90, 2002.
[11] F. Mainardi, *"On the initial value problem for the fractional diffusion-wave equation"*, Waves and Stability in Continuous Media, Bologna, 1993. Ser. Adv. Math. Appl. Sci., 23, 246-251, 1994.
[12] F. Mainardi and M. Tomirotti, *"On a special function arising in the time fractional diffusion-wave equation"*, Transform Methods and Special Functions, Sofia 1994. Rusev P., Dimovski I. and Kiryakova V. eds., Science Culture Technology, Singapore, 171-183, 1995.
[13] L. Vázquez, *"Fractional diffusion equation with internal degrees of freedom"*, J. Comp. Math., 21, n.4, 491-494, 2003.
[14] L. Vázquez and R. Vilela Mendes, *"Fractionally coupled solutions of the diffusion equation"*, Appl. Math. and Comp., 141, n.1, 125-130, 2003.
[15] A.A. Kilbas, T. Pierantozzi, J.J. Trujillo and L. Vázquez, *"On the solution of fractional evolution equation"*, J. Phys. A: Math. Gen., 37, 9, 3271-3283, 2004.
[16] T. Pierantozzi and L. Vázquez, *"An interpolation between the wave and diffusion equations through the fractional evolution equations* D*irac like"*, Journal of Mathematical Physics, 46, 113512, 2005.
[17] K. Diethelm, *"An algorithm for the numerical solution of differential equations of fractional order"*, Elect. Transact. Numer. Anal., 5, 1-6, 1997.
[18] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G Tricomi, *"Higher Transcendental Functions. Vol. 3"*, McGraw-Hill, New York, 1953.
[19] T. Pierantozzi and L. Vázquez, *"Stability bounds for some numerical schemes approximating fractional evolution-diffusion equations* D*irac like"*, Appl. Math. and Comp., 2006 (submitted).