©Civil-Comp Press, 2006.
Proceedings of the Fifth International Conference
on Engineering Computational Technology,
B.H.V. Topping, G. Montero and R. Montenegro,
(Editors), Civil-Comp Press, Stirlingshire, Scotland.

Paper 11

From a Fractional Mechanical Model to a Fractional Generalization of the Dirac Equation

D. Usero†, L. Vázquez‡# and T. Pierantozzi*
† Departmental Section of Applied Mathematics, Faculty of Chemical Sciences
‡ Department of Applied Mathematics, Faculty of Informatics
*** Department of Applied Mathematics, Faculty of Mathematics**
Universidad Complutense of Madrid, Spain
 # Centre for Astrobiology (CSIC-INTA), Torrejon de Ardoz, Madrid, Spain

Abstract

The Fractional Calculus represents a natural instrument to model non-local phenomena either in space or time that involve different scales. In this paper, we present a generalization of the linear one-dimensional diffusion and wave equations obtained by combining the fractional derivatives and the internal degrees of freedom associated to the system. Actually, taking into account that the free Dirac equation is, in some sense, the square root of the Klein-Gordon equation, in a similar way we can operate a kind of square root of the time fractional Diffusion equation in one space dimension through the system of fractional evolution-diffusion equations Dirac like. Solutions of the above system could model the diffusion of particles whose behavior depends not only on the space and time coordinates, but also on their internal structures. We study some analytical and invariance properties of the system highlighting its interpolation between the hyperbolic and parabolic behaviors.

Keywords: fractional differential equations, Riemann-Liouville fractional integrals and derivatives, Caputo fractional derivative, Mittag-Leffler and Wright functions, Dirac-type equations.

1 Introduction

The Fractional Calculus (see [13, 3], for example) represents a natural instrument to model nonlocal phenomena either in space or time that involve different scales. We present a generalization of the linear one-dimensional diffusion and wave equations by combining the fractional derivatives and the internal degrees of freedom associated to the system.

Following the well known Dirac's approach [14], it is possible to obtain his famous equation from the classical klein-Gordon equation. The free Dirac's equation is

$$A\frac{\partial\Psi}{\partial t} + B\frac{\partial\Psi}{\partial x} = 0 \tag{1}$$

where Ψ is a spinor, an structure that describes the state of the system and$A$ and B are matrices satisfying Pauli's algebra. In fact this equation can be considered as square root of Klein-Gordon wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

where u is a scalar field.

In a more general context Morinaga and Nono [10] analyzed the integer s-root of the partial differential equations of the form

$$\sum_{|I|=s} a_I \frac{\partial^{|I|}}{\partial x^I}\phi = \phi, \tag{3}$$

by defining them as the first order system

$$\sum_{i=1}^{n} \alpha_i \frac{\partial\Phi}{\partial x_i} = \Phi \tag{4}$$

where $\alpha_1, ...\alpha_n$ are matrices. From the physical point of view the α_k describe internal degrees of freedom of the associated system.

As can be suspected, this relationship between both equations results highly interesting to define roots of known operators. In the above context, Vázquez et al. recently considered in [19] and [20] the fractional diffusion equations with internal degrees of freedom. They can be obtained as the s-roots of the standard scalar linear diffusion equation. Thus, a possible definition of the square root of the standard diffusion equation (SDE) in one space dimension

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \tag{5}$$

is the following:

$$(A\frac{\partial^{1/2}}{\partial t^{1/2}} + B\frac{\partial}{\partial x})\psi(x,t) = 0 \tag{6}$$

where A and B are 2×2 matrices satisfying Pauli's algebra

$$A^2 = I, \quad B^2 = -I, \quad AB + BA = 0, \tag{7}$$

being I the identity operator.

Here $\psi(x,t)$ is a multicomponent function with at least two scalar space-time components. Also, each scalar component satisfies the SDE. Such solutions can be interpreted as *probability distributions with internal structure associated to internal degrees of freedom of the system.* They are named *diffunors* in analogy with the spinors in Quantum Mechanics.

We deal with a further generalization of Dirac's method. Concretely, we study the system of fractional evolution equations

$$A\frac{\partial^\alpha \Psi}{\partial t^\alpha} + B\frac{\partial \Psi}{\partial x} = 0, \quad \psi(t,x) = \begin{pmatrix} u_1(t,x) \\ u_2(t,x) \end{pmatrix}, \tag{8}$$

with $0 < \alpha < 1$ and where A and B are the same matrices as before satisfying Pauli's algebra.

System (8) generalize system (6) and was treated in [2] and [11] for the more general case of $0 < \alpha < 1$; if the index property ${\partial_t}^\alpha {\partial_t}^\alpha u = {\partial_t}^{2\alpha} u$ holds, it represents a decomposition of the Time Fractional Diffusion Equation (see for example [14], [7], [9], [4], [6]) in one space dimension

$$\partial_t^{2\alpha} u(t,x) - \partial_{xx} u(t,x) = 0, \tag{9}$$

through fractional equations of Dirac-type.

This decomposition was obtained operating a kind of square root of (9), as well as the free Dirac equation is considered, in some sense, as the square root of the Klein-Gordon equation [16].

Main characteristic of equation (8) is that for $\alpha \in (1/2, 1)$ allows a fractional interpolation in the sense of Dirac's between the classic wave equation ($\alpha = 1$) and the diffusion equation ($\alpha = 1/2$).

Equation (9)), associated to anomalous diffusion, has been widely studied in the literature by many authors, among which Schneider and Wyss [15], Metzler et al. ([7], [9], [21]), Mainardi et al. [4], [6], [5], Sokolov et al. [17], Saichev et al. [12]. A more physical discussion of this equation was given by Metzler and Klafter in [8] and it is worthwhile mentioning that they proved that the resulting probability density is bimodal in character. In this sense, Schneider and Wyss [15] showed that, for dimensions higher than 1, the character of the solution of the fractional wave equation (when $1/2 < \alpha \leq 1$) as a proper probability density is lost.

Each component of $\psi(t,x)$ satisfies (9) while the index property ${\partial_t}^\alpha {\partial_t}^\alpha u = {\partial_t}^{2\alpha} u$ holds. Thus, in the interval $1/2 < \alpha < 1$, the decomposition (8) of (9), expressed in terms of fractional evolution equations of Dirac-type, represents a fractional interpolation between the diffusion ($\alpha = 1/2$) and wave ($\alpha = 1$) equations.

Situation described above is summarized in Figure 1 where all the commented equations are represented with the existing relation between them. Also the fractional equations are placed interpolating the classic equations.

2 Physical meaning of the solutions of the system of fractional evolution equations

As can be observed in Figure 1, the Dirac's fractional diffusion equation (8) is closely related with the fractionary diffusion-wave equation

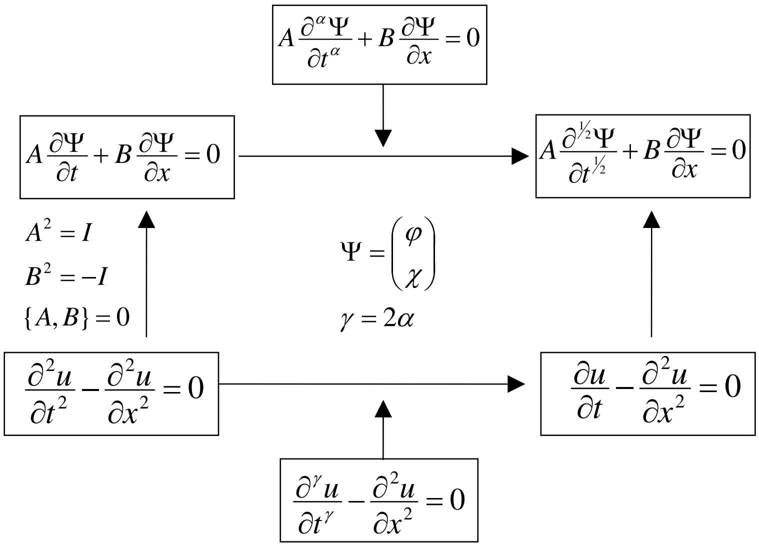

Figure 1: Scheme of the relations between different equations.

$$\partial_t^{2\alpha}u(t,x)-\partial_{xx}u(t,x)=0. \tag{10}$$

since first equation can be obtained from this one by taking the square root.

It is widely known that general solution of the wave equation with zero initial velocity

$$\left\{\begin{array}{c}\partial_{tt}u(t,x)-c^2\partial_{xx}u(t,x)=0\\ u(0,x)=\varphi(x),\quad \varphi\in\mathcal{C}^2\\ u_t(0,x)=0\end{array}\right., \tag{11}$$

is given by D'Alambert formula

$$u(t,x)=\frac{1}{2}\left[\varphi(x-ct)+\varphi(x+ct)\right], \tag{12}$$

where $\varphi(x\pm ct)$ are solutions of the two first order problems

$$\left\{\begin{array}{c}\partial_t u(t,x)-c\,\partial_x u(t,x)=0\\ u(0,x)=\varphi(x),\quad \varphi\in\mathcal{C}^2\end{array}\right., \tag{13}$$

and

$$\left\{\begin{array}{c}\partial_t u(t,x)+c\,\partial_x u(t,x)=0\\ u(0,x)=\varphi(x),\quad \varphi\in\mathcal{C}^2\end{array}\right.. \tag{14}$$

From a physical point of view, we can interpret this fact in the following way. The amplitude at a given time t of a perturbation created by a starting deformation at rest, $\varphi(x)$. This amplitude is the superposition of two waves, $\varphi(x+ct)$ and $\varphi(x-ct)$, whose shape is identical to the starting one's but traveling in opposite directions. The two waves are solutions of the first order problems (13) and (14) respectively.

Now, if we restrict our study to the pure real matrices leading to a system (8) of separated equations, then we have two possible choices:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{y} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{15}$$

and the second pair of matrices given by the same A and $-B$.

Substituting (15) in (8), it can easily be reduced to the following system of equations

$$\begin{cases} \partial_t^\alpha u_1(t,x) - \partial_x u_1(t,x) = 0 \\ \\ \partial_t^\alpha u_2(t,x) + \partial_x u_2(t,x) = 0 \end{cases}, \tag{16}$$

where $0 < \alpha < 1$, and the equations appearing in (13) and (14) can be recovered at the limiting case $\alpha = 1$ and $c = 1$.

The idea is to demonstrate that the solution $u(t,x)$ of the time fractional diffusion equation (9) is still a linear combination of the solutions $u_1(t,x)$ and $u_2(t,x)$ of (16), for each $0 < \alpha < 1$ as also happened in the case $\alpha = 1$ and, therefore, the relation existing between the Dirac solutions and the D'Alembert expression is valid also for the fractional case.

In order to get this objective it is necessary to recover some previous results.

First result concerns to fundamental solutions of equations like (16). In [2] the following initial value problem

$$\begin{aligned} &({}^C D_t^\alpha u)(t,x) = \lambda \partial_x u(t,x) \quad (t>0,\ x\in\mathbb{R};\ 0<\alpha<1), \\ &\lim_{|x|\to\infty} u(t,x) = 0, \quad u(0+,x) = g(x), \end{aligned} \tag{17}$$

has been solved and the general solution, expressed in terms of the inverse of its Fourier Transform, is given by

$$u(t,x) = u_{g,\lambda}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E_{\alpha,1}\left(-i\lambda k t^\alpha\right) G(k) e^{-ikx} dk, \tag{18}$$

where $G(k)$ is the Fourier transform of the initial condition $g(x)$ and $E_{\alpha,\beta}(z)$ is the biparametric Mittag-Leffler special function [1]. This solution is said to be *localized* due to the property $\lim_{|x|\to\infty} u(t,x) = 0$.

Another useful result is related with the fundamental solution of the fractional diffusion equation. The general localized solution of the Cauchy problem for the time fractional diffusion equation,

$$\begin{gathered} ({}^C D_t^{2\alpha} f)(t,x) = \lambda^2 \partial_{xx} f(t,x) \\ \lim_{|x|\to\infty} f(t,x) = 0, \quad f(0+,x) = g(x), \qquad [\partial_t f(t,x)]_{t=0} = 0, \quad (t>0,\ x\in\mathbb{R}), \end{gathered} \tag{19}$$

where $0 < \alpha < 1$, can be found in [5] and it reads

$$f(t,x) = f_g(t,x) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} E_{2\alpha,1}\left(-(\lambda k)^2 t^{2\alpha}\right) G(k) e^{-ikx} dk. \tag{20}$$

where $G(k)$ is again the Fourier transform of $g(x)$.

Having in mind equations (18) and (20) it is possible to apply the duplication formula [1] for the Mittag-Leffler function,

$$E_{2\alpha,1}(z) = \frac{1}{2}\left[E_{\alpha,1}(+z^{1/2}) + E_{\alpha,1}(-z^{1/2})\right], \tag{21}$$

to equation (20) and the it can be rewritten as follows:

$$f_g(t,x) = \frac{1}{2}\left[u_{g,-\lambda}(t,x) + u_{g,\lambda}(t,x)\right], \tag{22}$$

where $u_{g,\lambda}$ is given in (18).

So, we can conclude that the general solution of the Cauchy problem (19) for the time fractional diffusion equation turns out to be a linear combination, with coefficients equals to $1/2$, of the two general solutions of the Cauchy problems for the fractional Dirac-type equations represented by (17) and by the problem obtained when λ is replaced by $-\lambda$ in (17). D'Alambert formula is also valid for the fractionary case as we postulated before.

In particular, the fundamental solution of (19), when $g(x) = \delta(x)$, turns out to be

$$f(t,x) = \frac{1}{2\lambda\, t^{\alpha}} W\left(-\frac{|\, x\,|}{t^{\alpha}\lambda}; -\alpha, 1-\alpha\right), \quad (0 < \alpha < 1), \tag{23}$$

where $W(z;\alpha,\beta)$ is the Wright special function ([1, 18.1(27)]).

Therefore, if we assume $\lambda = 1$ in (19), we obtain

$$f(t,x) = \frac{1}{2t^{\alpha}} W\left(-\frac{|\, x\,|}{t^{\alpha}}; -\alpha, 1-\alpha\right) = \frac{1}{2}\left[u_1(t,x) + u_2(t,x)\right], \tag{24}$$

where

$$u_1(t,x) = \begin{cases} \frac{1}{t^{\alpha}} W\left(\frac{x}{t^{\alpha}}; -\alpha, 1-\alpha\right) & x \le 0, \\ 0 & x > 0, \end{cases} \tag{25}$$

and

$$u_2(t,x) = \begin{cases} 0 & x < 0, \\ \frac{1}{t^{\alpha}} W\left(-\frac{x}{t^{\alpha}}; -\alpha, 1-\alpha\right) & x \ge 0. \end{cases} \tag{26}$$

are the functions appearing in (16), fundamental solutions of (17) when $\lambda = 1$ and $\lambda = -1$, respectively (see [2]).

3 A Conserved Quantity: the Fractional Hamiltonian

It is well known (see, for example, [14]) that the Lagrangian density for the classical Dirac equation obtained from (8) when $\alpha = 1$, is given by

$$\mathcal{L}(t,x) = \bar{\psi} A \partial_t \psi + \bar{\psi} B \partial_x \psi,$$

with $\psi = \psi(t,x) = (u_1(t,x), u_2(t,x))^T$, $\bar{\psi} = \bar{\psi}(t,x) = \psi^+ A$, and $\psi^+ = \psi^+(t,x) = (u_1^*(t,x), u_2^*(t,x))$, complex conjugated of ψ that verifies the conjugated equation of (8) with $\alpha = 1$:

$$\partial_t \psi^+ A^+ + \partial_x \psi^+ B^+ = 0.$$

Therefore, the Hamiltonian density will be

$$\mathcal{H}(t,x) = \frac{\partial \mathcal{L}(t,x)}{\partial(\partial_t \psi)} \partial_t \psi - \mathcal{L}(t,x) = \bar{\psi} A \partial_t \psi - \mathcal{L}(t,x) = -\psi^+ C \partial_x \psi, \tag{27}$$

with $C = AB = -BA$, the Hamiltonian

$$H(t,x) = \int_{-\infty}^{+\infty} \mathcal{H}(t,x) dx = -\int_{-\infty}^{+\infty} \psi^+ C \partial_x \psi dx, \tag{28}$$

and, its time derivative,

$$\frac{d}{dt} H(t,x) = \int_{-\infty}^{+\infty} \partial_x \left[\psi^+ \partial_x \psi \right] dx, \tag{29}$$

provided that we restrict ourselves to the pure real matrices A and B so that the equivalence $\partial_t \psi^+ = -\partial_x \psi^+ C$ is verified. In this case, if we assume, for example, the initial condition $\psi^+ \partial_x \psi \to 0$ when $\mid x \mid \to \infty$, we can conclude that there exists a conserved quantity associated to equation (8) with $\alpha = 1$, given by the Hamiltonian (28).

Hamiltonian plays an important role in formulation of physical problems and in a previous work [18] a fractional hamiltonian formulation has been made for Caputo's time fractional derivative. In that work we consider Hamiltons equation where the time derivative is replaced by the Caputo fractional time derivative. In the Linear case, where and analytical solution is obtained, the fractional time derivative has the same effect as a linear damping force causing an asymptotic power law rather than exponential approach to the equilibrium. For non linear equations a similar behavior is observed, in agreement with a quadratic approximation of the potential near its minima.

In what follows, we want to show that, as well as we did above, it is possible to find a conserved quantity associated to the system of fractional Dirac-type equations (8) for general $0 < \alpha < 1$, even if it does not present invariance with respect to time translation.

We start defining, by analogy with the classical Dirac case, a formal "fractional Lagrangian density" related to (8)

$$\mathcal{L}^{\alpha}(t,x) = \bar{\psi} A \partial_t^{\alpha} \psi + \bar{\psi} B \partial_x \psi, \tag{30}$$

and a formal "fractional Hamiltonian density"

$$\mathcal{H}_{\alpha}(t,x) = \frac{\partial \mathcal{L}^{\alpha}(t,x)}{\partial(\partial_t^{\alpha}\psi)} \partial_t^{\alpha}\psi - \mathcal{L}^{\alpha}(t,x) = \bar{\psi} A \partial_t^{\alpha}\psi - \mathcal{L}^{\alpha}(t,x) = -\psi^{+} C \partial_x \psi. \tag{31}$$

The final expression in (31) is equivalent to (27) and can be simplified observing that, being A and B pure real matrices, if ψ is a solution of (8), then also ψ^{+} has to solve it, reason why a pure real solution of (8) can always be found. Therefore, we will assume that ψ it is a pure real solution of (8) and this allows us to write the "fractional Hamiltonian" as

$$H_{\alpha}(t,x) = \int_{-\infty}^{+\infty} \mathcal{H}_{\alpha}(t,x) dx = -\int_{-\infty}^{+\infty} \psi^{T} C \partial_x \psi dx \tag{32}$$

and, consequently,

$$\frac{d}{dt} H_{\alpha}(t,x) = -\int_{-\infty}^{+\infty} \left[\partial_t \psi^{T} C \partial_x \psi + \psi^{T} C \partial_x \partial_t \psi \right] dx = -\int_{-\infty}^{+\infty} \partial_x \left[\psi^{T} C \partial_t \psi \right] dx. \tag{33}$$

It has been used the equivalence $\partial_t \psi^{T} C \partial_x \psi = \partial_x \psi^{T} C \partial_t \psi$, due to the fact that the matrix $C = AB$, with pure real A and B, only can be of two types:

$$C_1 = \begin{pmatrix} c_{11} & 0 \\ 0 & -c_{11} \end{pmatrix} \quad \text{o} \quad C_2 = \begin{pmatrix} 0 & c_{12} \\ c_{12} & 0 \end{pmatrix},$$

where c_{11} and c_{12} take the values ± 1.

At this point it is necessary to specify the definition of the fractional derivative in use.

When $0 < \alpha < 1$ the Riemann-Liouville fractional derivative $\partial_t{}^{\alpha} = {}^{RL}D_t^{\alpha}$ fulfils the following identity

$$({}^{RL}D_t^{\alpha}\psi)(t,x) = (I_t^{1-\alpha}\partial_t\psi)(t,x) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\psi(0,x), \tag{34}$$

being $({}^{RL}D_t^{\alpha}\psi(0,x))(t,x) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\psi(0,x)$ and $({}^{C}D_t^{\alpha}\psi)(t,x) = (I_t^{1-\alpha}\partial_t\psi)(t,x)$ by definition of the Caputo's fractional derivative.

Now, if we introduce the Riemann-Liouville fractional derivative in (8), using the fact that this derivative is the left inverse operator of the Riemann-Liouville fractional integral, we can write

$$(\partial_t\psi)(t,x) = -C\partial_x({}^{RL}D_t^{1-\alpha}\psi)(t,x) - A({}^{RL}D_t^{1-\alpha}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\psi(0,x))(t,x) =$$

$$= -C\partial_x({}^{RL}D_t^{1-\alpha}\psi)(t,x) - A\partial_t\psi(0,x) = -C\partial_x({}^{RL}D_t^{1-\alpha}\psi)(t,x). \qquad (35)$$

In a similar straightforward way it can be proved that result (35) holds exactly the same when the Caputo derivative appears in (8).

Therefore, when $\alpha = 1/2$ both derivatives verify

$$(\partial_t\psi)(t,x) = C^2(\partial_x^2\psi)(t,x) = (\partial_x^2\psi)(t,x). \qquad (36)$$

In agreement with (35), the expression for the Hamiltonian time derivative (33) takes the form:

$$\frac{d}{dt}H_\alpha(t,x) = -\int_{-\infty}^{+\infty} \partial_x\left[\psi^T C\partial_t\psi\right]dx = \int_{-\infty}^{+\infty} \partial_x\left[\psi^T\,\partial_x\,{}^{RL}D_t^{1-\alpha}\psi\right], \qquad (37)$$

when $0 < \alpha < 1$ and, in particular,

$$\frac{d}{dt}H_{1/2}(t,x) = \int_{-\infty}^{+\infty} -\partial_x\left[\psi^T\,C\,\partial_x^2\,\psi\right]dx, \qquad (38)$$

for $\alpha = 1/2$, when the fractional derivative is either of the Riemann-Liouville or of the Caputo type.

Therefore, we can conclude that, when $0 < \alpha < 1$, if the condition

$$\frac{d}{dt}H_\alpha(t,x) = \int_{-\infty}^{+\infty} \partial_x\left[\psi^T\,\partial_x\,{}^{RL}D_t^{1-\alpha}\psi\right]dx =$$

$$= \left[u_1\,\partial_x\,{}^{RL}D_t^{1-\alpha}u_1 + u_2\,\partial_x\,{}^{RL}D_t^{1-\alpha}u_2\right]\Big|_{x=-\infty}^{x=+\infty} = 0, \qquad (39)$$

is fulfilled, then a conserved quantity exists associated to equation (8) and it is given by the fractional Hamiltonian (32).

4 Conclusions

In the present work we have treated a generalization of the classical free Dirac equations called *Dirac-type fractional evolution equations*. We have shown the relation between this equations and the well known *fractional diffusion equation* showing a panoramic review of all this equations and the relationship between them.

We have also shown that D'Alambert formula can be generalized to the solution of the *fractional diffusion equation* since this solution is a linear combination of the solutions of Dirac-type fractional evolution equations.

Following a mechanical formulation we have also obtained a conserved quantity, the fractional Hamiltonian, analogous to the classical one.

References

[1] A. Erdelyi, W. Magnus, F. Oberhettinger, F. Tricomi,*"G. Higher Transcendental Functions"*, Vol.3, McGraw-Hill Book Corp., New York, 1953.

[2] A.A. Kilbas , T. Pierantozzi, J. Trujillo, L. Vázquez ,*"On the solution of fractional evolution equations"*, Journal of Physics A: Mathematical and General, 37, 3271-3283, 2004.

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *" Theory and Applications of Fractional Differential Equations"*,Elsevier, 2006.

[4] F. Mainardi, *"On the initial value problem for the fractional diffusion-wave equation, Waves and Stability in Continuous Media"*, Bologna, 1993, *Ser. Adv. Math. Appl. Sci.*, n. 23, 246-251, 1994.

[5] F. Mainardi, P. Paradisi, R. Gorenflo *"Probability distributions generated by fractional diffusion equations"*, in *"Econophysics: an Emerging Science"*, Kertesz J. and Kondor I. eds., Dordrecht: Kluwer, 1999.

[6] F. Mainardi, M. Tomirotti, *"On a special function arising in the time fractional diffusion-wave equation, Transform Methods and Special Functions"*, Sofia 1994. Rusev P., Dimovski I. and Kiryakova V. eds., Science Culture Technology, Singapore, 171-183, 1995.

[7] R. Metzler, J. Klafter,*"The Random Walk's Guide to Anomalous Diffusion: A Fractional Dynamics Approach"*, Phys. Reports, 339, 1-77, 2000.

[8] R. Metzler, J. Klafter, *"Accelerating Brownian motion: a fractional dynamics approach to fast diffusion"*, Europhys. Lett., 51, 492-498, 2000.

[9] R. Metzler, T.F. Nonnemacher,*"Space- and Time-fractional diffusion and wave equations, fractional Fokker-Planck equations, and physical motivation"*, Chem. Phys., 284, n. 1-2, 67-90, 2002.

[10] K. Morinaga, T. Nono, J. Sci. Hiroshima Univ. (A), 16, 13-41, 1952.

[11] T. Pierantozzi, L. Vázquez, *"An interpolation between the wave and diffusion equations through the fractional evolution equations Dirac like"*. Journal of Mathematical Physics, 46, 113512, 2005.

[12] A. I. Saichev, G. M. Zaslavsky, *"Fractional kinetic equations: solutions and applications"*, Chaos, 7, 753-764, 1997.

[13] S. G. Samko, A.A. Kilbas , O. I. Marichev, *"Fractional Integrals and Derivatives. Theory and Applications"*, Gordon and Breach, Yverdon, 1993.

[14] W. R. Schneider, W. Wyss, *"Fractional diffusion and wave equations"*, J. Math. Phys., 30, 134-144, 1989.

[15] W. R. Schneider,W. Wyss , *"Fractional diffusion and wave equations"*, J. Math. Phys., 30, 134-144, 1989.

[16] S. S. Schweber, *"An Introduction to Relativistic Quantum Field Theory"*, Harper Int. Ed. New York, 1966.

[17] I. Sokolov, J. Klafter, A. Blumen *"Fractional Kinetics"*, Physics Today, 55,48-54, 2002.

[18] G. Turchetti, D. Usero, L. Vázquez, *"Hamiltonian Systems with Fractional Time Derivative"*, Tamsui Oxford Jounal of Mathematical Physics, 18-1, 31-44, 2002.

[19] L. Vázquez, *"Fractional diffusion equation with internal degrees of freedom"*, J. Comp. Math., Vol. 21, n. 4, 491-494, 2003.

[20] L. Vázquez, R. Vilela Mendes, *"Fractionally coupled solutions of the diffusion equation"*, Applied Mathematics and Computation, 141, 125-130, 2003.

[21] M. O. Vlad , R. Metzler, T. F. Nonnenmacher, M. C. Mackey, *"Universality classes for asymptotic behavior of relaxation processes in systems with dynamical disorder: Dynamical generalizations of stretched exponential"*, J. Math. Phys., 37, 2279-2306, 1996.