Operators of Generalized Fractional Calculus & Special Functions of Fractional Calculus – Closely interrelated

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In this lecture we survey in brief the ideas of the generalized fractional calculus (GFC, [1]) and its close interrelation with the Special Functions (SF). Here, under the term GFC we mean integral, differ-integral and integro-differential operators with singularities which involve SF in the kernels, have convolutional structure and satisfy the basic "axioms" of the classical FC. As their particular cases, they appear the operators of classical FC of Riemann-Liouville and Caputo type, Erdelyi-Kober (E-K) operators and their compositions named as Saigo, Marichev-Saigo-Maeda, Hyper-Bessel operators as well as many other operators of generalized integration and differentiation studied and used in Calculus, Differential and Integral Equations and their applications in mathematical models.

The notion "generalized operators of fractional integration" appeared in the papers of Kalla in the years 1969-1979, see [2]. He suggested the general form of these operators, studied some of their formal properties and examples of them when the kernels were special functions as the Gauss and generalized hypergeometric functions, including arbitrary *G*- and *H*-functions. His ideas provoked the author to choose a more peculiar case of *G*- and *H*-kernels ($G^{m,n}_{m+n,m+n}$ or $H^{m,n}_{m+n,m+n}$) which allowed developing a full theory of the corresponding GFC ([1]) with many particular cases and applications.

On the other side, in Samko-Kilbas-Marichev [3], for analytic functions we can find the notion of generalized differentiation and integration defined by Hadamard products with suitable multipliers' sequences. And specifically, when such a sequence is formed by the quotients of subsequent coefficients of an entire function, these are the Gel'fond-Leont'ev (G-L) operators of 1951, [4]. Taking the entire function to be either the exponential function, the Mittag-Leffler (M-L) function or other special functions of FC, we obtain the classical integration and differentiation, the operators of classical FC and the operators of our GFC, and all their particular cases as mentioned, see e.g. [5]. Important role is played by the Wright generalized hypergeometric functions and the multi-index M-L functions, [6] and [7]. All these SF appear in the explicit solutions of various differential and integral equations of high integer or fractional order (or multi-order), and also as eigen functions of the involved differential and integral operators. In series of papers, we have emphasized on interesting illustrative examples.

Recently many authors are spending lot of time and efforts to evaluate various operators of FC of quite particular SF or of classes of SF. The list of such works, that can be considered also as evaluation improper integrals of products of some elementary and special functions, is rather long and is yet growing daily. Since the special functions present indeed a great variety, and the operators of fractional calculus do as well, the mentioned job produces a huge flood of publications. Many of them use same formal and standard procedures, and besides, often the results sound not of practical use, with except to increase authors' publication activities. Our approach, based on the ideas of GFC provides a simple and unified way to do such task at once, in the rather general case: for both operators of generalized fractional calculus and for the generalized hypergeometric functions $p\Psi q$ (as general SF of FC). Thus, great part of the results in the mentioned publications are well predicted and fall just as particular cases of our general scheme, see [8].

Refs: [1] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman and J. Wiley, 1994; [2] S.L. Kalla, Operators of fractional integration, In: *Lecture Notes in Math.* **798** (1980), 258-280; [3] S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon & Breach. Sci. Publ., 1993; [4] A.O. Gel'fond, A.F. Leont'ev, On a generalization of the Fourier series (in Russian), *Mat. Sbornik* **29**, No 71 (1951), 477-500; [5] V. Kiryakova, Gel'fond-Leont'ev integration operators of fractional (multi-)order generated by some special functions, *AIP Conf. Proc.* (AMEE 2018), 2018, to appear; [6] V. Kiryakova, The special functions of fractional calculus as generalized fractional calculus operators of some basic functions, *Computers and Math. with Appl.* **59**, No 3 (2010), 1128-1141; [7] V. Kiryakova, The multi-index Mittag-Leffler functions as important class of special functions of fractional calculus, *Computers and Math. with Appl.* **59**, No 5 (2010), 1885-1895; [8] V. Kiryakova, Fractional calculus operators of special functions? The result is well predictable! *Chaos, Solitons and Fractals* **102** (2017), 2-15; etc.