FRACTIONAL INTEGRALS WITH ORDER VARYING IN TIME FOR IMAGE PROCESSING: LONGTIME BEHAVIOR OF THE CONTINUOUS SOLUTIONS AND ITS TIME DISCRETIZATIONS

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Abstract

In this work we consider the Volterra type vector equation

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t AD(t-s)\mathbf{u}(s)\,\mathrm{d}s, \quad t>0, \tag{1}$$

where,

- The initial data $\mathbf{u}_0 \in \mathcal{M}_{J^2\times 1}(\mathbb{R})$ stands for a vector whose inputs correspond to the pixels of a gray–scale image uniform $J\times J$ sampling.
- The vector function $\mathbf{u} : (0,T] \to \mathcal{M}_{J^2\times 1}(\mathbb{R})$, stands for the evolved initial data $\mathbf{u}_0$ (original image) up to the time level $t>0$, in fact $\mathbf{u}(t) = [u_j(t)]^T_{1\le j\le J^2}$.
- The matrix function $D : (0,T] \to \mathcal{M}_{J^2\times J^2}(\mathbb{R})$, is here considered as a diagonal matrix $D = \mathrm{diag}_{1\le j\le J^2}\{k_j(t)\}$.
- $A \in \mathcal{M}_{J^2\times J^2}(\mathbb{R})$ is a symmetric and negative semi–definite matrix.

In particular, the equation (1) is of fractional type since each single kernel k_j defines the fractional integral with order $\alpha_j(t)>0$, $t\ge 0$,

$$\partial^{-\alpha_j(t)} g(t) := \int_0^t k_j(t-s) g(s)\,\mathrm{d}s, \quad t>0, \tag{2}$$

for $g \in L^1((0,T],\mathbb{R})$. In our talk we will discuss on the suitability of one definition of fractional integral with order varying in time vs. the other ones existing in the literature.

On the other hand, the matrix A in the equation (1) typically arises (but not only!) from the spatial discretization of a differential operator, e.g. the Laplacian operator, which so many times gives rise to a symmetric and negative semi–definite matrix.

This model was introduced, studied, and successfully applied to image filtering in [1], where among this model more general nonlocal in time equations were considered.

Recently, the scalar version of (1) namely

$$u(t) = u_0 + \int_0^t k(t-s)Au(s)\,\mathrm{d}s, \quad t>0, \tag{3}$$

where $k(t)$ is of the type mentioned above, and A is an unbounded linear operator, has been accurately studied. In fact, well–posedness, asymptotic behavior, regularity at $t=0$, and the counterpart properties in the framework of time discretizations have been proved (see [2]).

The interest of these results arises in the fact that one of the most important properties in the context of mathematical models for image processing is the asymptotic behavior of their solutions since this fact helps to determine the restoration quality can be reached by means of such a models. The present work extends the results achieved for scalar equations of type (3) in [2], where an abstract and very general formulation in the setting of Banach spaces is considered, to systems of type (1), and some of its time discretizations.

References

[1] E. Cuesta, A. Durán, and M. Kirane, *On evolutionary integral models for image restoration.* Developments in Medical Image Processing and Computational Vision, 241-260, 2015.

[2] E. Cuesta, and R. Ponce, *Well–posedness, regularity, and asymptotic behavior of the continuous and discrete solutions of linear fractional differential equations with order varying in time.* (submitted)

Acknowledgments: This work was supported by Department of Applied Mathematics, University of Valladolid.

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